International Research Journal of Applied and Basic Sciences
© 2013 Available online at www.irjabs.com
ISSN 2251-838X / Vol, 4 (8): 2375-2380
Science Explorer Publications

Some properties of hyperring

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ABSTRACT: In this paper, some properties of intersection, union, sum and product hyperring are discussed. Generally, intersection family of hyperideals of a hyperring is hyperideal, but, union family of hyperideals of a hyperring is not hyperideal. Also the property is hyperideal in the subhyperrings of a hyperring, with no transition. We give condition such that this property is true.
***Keywords**:* Cononical hypergroup, Hyperring, Regular hyperring, Hyper-ideal, Sum hyperideals, Product hyperideals.

INTRODUCTION

The theory of algebraic hyperstructures which is a generalization of ordinary algebraic structures was first introduced by Marty(Kuroki,1981). Since then, many researchers have studied the theory of hyperstructures and developed it. Moreover, the applications of this theory in other fields such as geometry, graphs and hypergraphs, lattices, automata, cryptography, codes, etc has been extensively studied, see(Asghari,2012; Asghari,2009;Asghari,2010;Corsini,1993; Marty,1934; Ranjbar,2011).

Krasner has studied the notion of hyperringin (Davvaz,2009). Davvaz has defined some relations in hyperrings and prove isomorphism theorems in (Corsini,1993). This paper is structured as follows. After the introduction, in section 2, we recall some basic notions and results on canonical hypergroup. In section 3, we study some properties of hyperring, give several examples and we establish some characterization theorems. Finally, in Section 4, several characterization theorems of kranser hyperring are obtained, particularly about the relation between hyperring and sum direct sum and product hyperideals.

2 Preliminaries

We recall some definitions (Corsini,1993), which we need in what follows.

Definition 2.1. Let H be a non-empty set and $P^*(H)$ be the family of all non-empty subsets of H. A hyperoperation or join operationis a map $+ : H \times H \to P^*(H)$. If $(a, b) \in H \times H$, then its image under " $+$ " is denoted by a+b.

The join operation is extended to subsets of H in a natural way, so that $A + B$ is given by

$$A + B = \cup \{a + b | a \in A, b \in B\}.$$

The notations $a + A$ and $A + a$ are used for $\{a\} + A$ and $A + \{a\}$ respectively.

Generally, the singleton {a} is identified by its element a.

A non-empty set H, endowed with a hyperoperation " $+$ " is called a hypergroupoid and it is denoted by $(H, +)$. If

$$x + (y + z) = (x + y) + z, \forall x, y, z \in H$$

then $(H,+)$ is called a semihypergroup.

A hypergroupoid $(H, +)$ is called a quasihypergroup, if $x + H = H = H + x$,
for all $x \in H$.

Definition 2.2. A hypergroup is a semihypergroup and a quasihypergroup.

Let $(H, +)$ be a hypergroupoid and K a non-empty subset of H, then K is called a subhypergroupoid of H, such that $K + K \subseteq K$. A subhypergroupoid K of H is called a subhypergroup of H, if $(K, +)$ is a hypergroup.

Definition 2.3.A hypergroup $(H, +)$ is called a canonical hypergroup if the following conditions are satisfied:

(i) $x + y = y + x, \forall x, y \in H$

(ii) $\exists 0 \in H$(unique)such that $0 + x = x = x + 0, \forall x \in H$

(iii) $\forall x \in H$ there exists an unique element $- x \in H$ such that $0 \in x + (-x)$

(iv) $\forall x, y, z \in H, z \in x + y \to x \in z + (-y)$and $y \in z + (-x)$

A subhypergroup K of a canonical hypergroup $(H,+)$ is called a canonical subhypergroup of H, if it is a canonical hypergroup with respect to the hyperoperation "+" of H.

Example 2.4.(1) Let $H = \{0,1,2,3,4,5\}$ be the cyclic group Z_6 and $x + y = L\{x,y\}$, where $L\{x,y\}$ is the subgroup of H generated by $\{x,y\}$ .Then the hypergroupoid $(H,+)$ defined by the following table:

Tabel 1: The hypergroupoid

+	0	1	2	3	4	5
0	{0}	H	{0,2,4}	{0,3}	{0,2,4}	H
1	H	H	H	H	H	H
2	{0,2,4	H	{0,2,4}	H	{0,2,4}	H
3	{0,3}	H	H	{0,3}	H	H
4	{0,2,4}	H	{0,2,4}	H	{0,2,4}	H
5	H	H	H	H	H	H

is a commutative hypergroup. Clearly $(H,+)$ is not a canonical hypergroup.

(2) Let $H = \{(a_{ij})_{n\times n}: a_{ij} \in \mathbb{R}, \forall i \neq j, a_{ij} = 0, \det(a_{ij}) \neq 0\}$. We define the hyperoperation "$\circ$" as:

$$\circ(A,B) = A \circ B = \{AB\} \text{ for all } A, B \in H$$

Then hypergroupoid $(H, \circ)$ is a hypergroup, and

$$\text{(i)} A \circ B = \{AB\} = \{BA\} = B \circ A, \qquad \text{for all } A, B \in H$$

(ii) Multiplicative identity matrix I_n is an identity elements of H.

(iii) If $A \in H$, then $I_n = AA^{-1} \in \{AA^{-1}\} = A \circ A^{-1}$. Thus A^{-1} is inverse of A. Clearly A^{-1} is unique.

(iv) $(H, \circ)$ is reversible, because $B \in A \circ C = \{AC\}$. Then $B = AC$. Consequently, $C = A^{-1}B \in \{A^{-1}B\} = A^{-1} \circ B$ and $A = BC^{-1} \in \{BC^{-1}\} = BC^{-1}$. Thus $(H, \circ)$is a canonical hypergroup.

(3) Let $H = \{0,1,2,3\}$ be a set with the hyperoperation " + " defined as follow:

Tabel 2: The canonical hypergroup

+	0	1	2	3
0	0	1	2	3
1	1	1	H	1
2	2	H	2	2
3	3	1	2	0

Then $(H,+)$ is a canonical hypergroup.

3 Some properties of hyperring and regular hyperring

Definition 3.1. (Corsini,1993]) A hyperring is an algebraic structure $(R,+,.)$, where
R is a non-empty set with a hyperadition " + " and a hypermultiplication "."
which satisfies the following axioms:

(i) $(R,+)$ is a canonical hypergroup;

(ii) $(R,.)$ is a semihypergroup haveing zero as a bilaterally absorbing element, i.e., $x.0 = 0.x = 0$ for all $x \in R$;

(iii) The hypermultiplication "." is distributive with respect to the hyperoperation " + ". That is for every $x,y,z \in R$, $z.(x+y) = z.x + z.y$,and $(x+y).z = x.z + y.z$

In a hyperring if the hypermultiplication is a binary operation, then it is called as kransker or additive hyperring. Also, in the hyperaddition is a binary operation, then it is called as multiplicative hyperring.

The following elementary facts follow easily from the axioms: $-(-x) = x$and $-(x+y) = -x-y$, where $-A = \{-a: a \in A\}$.

A hyperring $(R,+,.)$ is called commutative if $(R,.)$ is a commutative.

A hyperring $(R,+,.)$ is called hyprring with identy if $1_R \in R$ such that $x.1_R = 1_R.x = x$ for all $x \in R$.

Definition 3.2. (Corsini,1993) If $(R,+,.)$ is a hyperring and H is a non-empty subset of R, we say that $(H,+,.)$ is a subhyperring of R if $(H,+)$ is a canonical subhypergroup of $(R,+)$ and $(H,.)$ is a subsemigroup of $(R,.)$.

Example 3.3.Let $H = \{0,1,2,3\}$ and the hyperoperation " + " is defined in Example 2.4(6) . If the hypermultiplication "." defined as follows:

Tabel 3: The hyperring

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	3
3	0	0	0	0

Then $(\mathrm{H},+,.)$ is a hyperring.

Example 3.4.Let $\mathrm{R}=\{(a_{ij})_{n\times n}: a_{ij}\in\mathbb{R}\}$, we define the hyperoperation " + " as: $A+B=\{A+B\}$. Then $(\mathrm{R},+,.)$ where operation "." is binary multiplication will be a krasner hyperring with additive identity as a null matrix and multiplicative identity I_n

Lemma 3.5.Let H_1 and H_2 are canonical subhypergroup of $(H,+)$. If $a\in H_1\cap H_2$, then there exists a unique $(-a)\in H_1\cap H_2$, such that $0\in a+(-a)$.

Proof. Since $a\in H_1\cap H_2$, thus $a\in H_1$ and $a\in H_2$. Since $a\in H_1$, so there exists a unique $b\in H_1$ such that $0\in a+b$. Similarly, there exists a unique $c\in H_2$ such that $0\in a+c$. Since H is reversible, thus $b\in 0+(-a)=\{-a\}$ and $c\in 0+(-a)=\{-a\}$. Which implies b=c . Hence for every $a\in H_1\cap H_2$ there exists a unique $(-a)\in H_1\cap H_2$, such that $a\in a+(-a)$.

Theorem 3.6. Let $(R,+,.)$ is a hyperring and $\{H_i\}_{i\in I}$ be a family of subhyperring of R. Then $\cap_{i\in I} H_i$ is also a subhyperring or R.

Proof. Since for all $i\in I, 0\in H_i$, so $0\in\cap_{i\in I} H_i$. Thus $\cap_{i\in I} H_i\neq\emptyset$. Let $a,b\in\cap_{i\in I} H_i$ then $a+b\subseteq H_i$for all $i\in I$. Since each H_i is a subhyperring so $a+b\subseteq\cap_{i\in I} H_i$. Clearly $(\cap_{i\in I} H_i,+)$ is a commutative subhypergroup of $(R,+)$ and for every $a\in\cap_{i\in I} H_i, a+0=\{a\}$.

By Lemma 3.5, If $x\in\cap_{i\in I} H_i$ then there exists a unique $y\in\cap_{i\in I} H_i$ such that $0\in x+y=y+x$. Also $(\cap_{i\in I} H_i,+)$ is reversible. Thus $(\cap_{i\in I} H_i,+)$ is a canonical subhypergroup $(R,+)$. On the other hand, if, $a,b\in\cap_{i\in I} H_i$ then $a.b\subseteq\cap_{i\in I} H_i$ (since for all $i\in I$, H_i is a subhyperring) and $(\cap_{i\in I} H_i,.)$ is subsemigroup of $(R,.)$. Hence $\cap_{i\in I} H_i$is also a subhyperring of R.

Definition 3.7. (Corsini,1993) If $(R,+,.)$ be a hyperring and I is a non-empty subset of R, we say that $(I,+,.)$ is a left (resp. right) hyperideal of R, if $(I,+)$ is a canonical subhypergroup of R and $R.I\subseteq I$ (resp.$I.R\subseteq I$).

Similarly, we can define the notion of two-sided hyperideal of R.

Example 3.8. Let $R=\{\begin{pmatrix}a&0\\b&c\end{pmatrix}: a,b,c\in\mathbb{Z}\}$.Then $(R,+,.)$ defined in example of 3.4 is a hyperring. If $I=\{\begin{pmatrix}a&0\\b&0\end{pmatrix}: a,b\in\mathbb{Z}\}$. Then I is a left hyperideal of R but is not a right hyperideal of R. Because $\begin{pmatrix}1&0\\0&0\end{pmatrix}\in I$ and $\begin{pmatrix}0&1\\0&0\end{pmatrix}\in R$, but $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\notin I$.

Example 3.9.Let $R=\{(a_{ij})_{n\times n}: a_{ij}\in\mathbb{R},\forall i\neq j, a_{ij}=0\}$. Then $(R,+,.)$ defined in example of 3.4 is a commutative hyperring with identity I_n. If $I_i=\{a.E_{ii}: a\in\mathbb{R}\}$ where E_{ii} is the matrix with ij-entry 1 and 0 elsewhere. Then for all i, $I_i$is a tow-sided hyperideal of R.

Theorem 3.10. Let $(R,+,.)$ is a hyperring and $\{I_i\}_{i\in s}$ be a family of hyperideal of R. Then $\cap_{i\in S} I_i$ is also a hyperideal of R.

proof. Since for all $i\in S$, I_i is a subhyperring of R, by theorem of 3.6, $\cap_{i\in S} I_i$ is a subhyperring of R. Suppose $x\in R$ and $a\in\cap_{i\in S} I_i$, implies $a\in I_i$ for all i ∈S, and I_i is a hyperideal, so $x.a\in I_i$ for all $i\in S$. Therefore $x.a\in\cap_{i\in S} I_i$. Similarly, we can show that $ax\in\cap_{i\in S} I_i$. Hence $\cap_{i\in S} I_i$ is a hyperideal of a hyperring R.

Remark 3.11. Let I and J be hyperideals of a hyperring R. If I ∩ J={0}, then for all $a\in I$ and , $b\in J$, a.b={0}.

Proof.Straight forward.

Example3.12.Let $R=\left\{\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\0&0&a_{33}\end{pmatrix}: a_{ij}\in\mathbb{R}, I=1,2,3, J=1,2,3\right\}$ and $I=\left\{\begin{pmatrix}0&0&a\\0&0&0\\0&0&0\end{pmatrix}: a\in\mathbb{R}\right\}$, $J=\left\{\begin{pmatrix}0&0&a\\0&0&b\\0&0&0\end{pmatrix}: a\in\mathbb{R}\right\}$. Then R with operation defined in example of 3.4 is a Krasner hyperring and I is a hyperideal of J and J is a hyperideal of R but I is not a hyperideal of R because,

$A=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\in R$ and $\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\in I$, but $A.B=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\notin I$.

Corollary 3.13. In general, if I be a hyperideal of J and J be a hyperideal R then I is not a hyperideal of R.

Definition 3.14.We say that an element x in a multiplicative hyperring $(R,+,.)$ is regular if $x \in x.y.z$ for some $y \in R$. Hyperring R is called a regular hyperring if every element of R is regular.
Theorem 3.15. Let $(R,+,.)$ be a regular multiplicative hyperring. If I be
A hyperideal of J and J be a hyperideal of R, then I is a hyperideal of R.
Proof. Since I is a subhyperring of R, thus we need only prove that for every $a \in I$ and $x \in R, x.a \subseteq I$ and $a.x \subseteq I$.
If $x \in r.a$, then $\exists y \in R$ such that $x \in x.y.x$. Show that $x.y.z \subseteq (r.a).y.(r.a)$. Since $x \in r.a$, so $x.y \subseteq (r.a).y$, then $x.y.x = (x.y).x \subseteq [(r.a).y].(r.a)$. Hence
$x \in (r.a).y.(r.a)$ (1)
Since J is a hyperideal of R and $a \in J$, thus $r.a \subseteq J$, so $[(r.a).y].r \subseteq J$. Thus $(r.a).y.r \subseteq J$. Since I is a hyperideal of J and $a \in I$ thus
$(r.a).y.(r.a) = [(r.a).(y.r)].a \subseteq I$ (2)
Thus from (1) and (2) we obtain $x \in I$. Hence $r.a \subseteq I$. Similarly, we can show that $a.r \subseteq I$. Therefore I is a hyperideal of R.
Example 3.16. In example of 3.4, let $H = \{A \in R: A$ is a upper triangular matrix$\}$ and $K = \{A \in R: A$ is a lower triangular matrix$\}$ then, H, K are subhypergrrings of R. Also, $E_{1n} \in H$ and $E_{n1} \in K$ but,

$$E_{1n} + E_{n1} = \begin{pmatrix} 0 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 0 \end{pmatrix} \nsubseteq H \cup K$$

Corollary 3.17. In general, union of two subhyperring of a hyperring is not a subhyperring.
Theorem 3.18. Let H, K be subhyperings of a hyperring R. Then $H \cup K$
is a subhyperring R, if and only if $H \subseteq K$ or $K \subseteq H$.
Proof.Straight forward.

4 Sum and product hyperideals in kranser hyperring

Lemma 4.1.(Davvaz,2009) A nonempty subset I of a kranser hyperring R is a left (resp. right) hyperideal if and only if
(i) $a, b \in I$ implies $a - b \subseteq I$
(ii) $a \in I, r \in R$ implies $r.a \in I$ (resp. $a.r \in I$)
Definition 4.2. (Davvaz,2009) Let I and J be nonempty subsets of a kranser hyperring R.
(i) The sum $I + J$ is defined by

$$I + J = \{x: x \in a + b \text{ for some } a \in I, b \in J\}$$

(ii) The product IJ is defined by

$$IJ = \{x: x \in \sum_{i=1}^{n} a_i b_i , a_i \in I, b_i \in J, n \in \mathbb{Z}^+\}$$

If I and J are hyperideals of R, then $I + J$ and IJ are also hyperideals of R.
Definition 4.3. Let $I_1, I_2, \dots I_n$ be hyperideals of a hyperring R. Then we define the sum $\sum_{i=1}^{n} I_i$ and the product $\prod_{i=1}^{n} I_i$ as:

$$\sum_{i=1}^{n} I_i = \bigcup_{a_i \in I} \sum_{i=1}^{n} a_i , \prod_{i=1}^{n} I_i = \bigcup_{a_{ij} \in I} \sum_{\text{finite}} a_{1j} a_{2j} \dots a_{nj}$$

Theorem 4.4. Let $I_1, I_2, \dots I_n$be hyperideals of hyperring R. Then $\sum_{i=1}^{n} a_i$and $\prod_{i=1}^{n} I_i$ are hyperideals of R.
Proof. Let $x, y \in \sum_{i=1}^{n} I_i$ then for all i there exist $a_i, b_i \in I_i$ such that $x \in \sum_{i=1}^{n} a_i$, $y \in \sum_{i=1}^{n} b_i$ thus

$$x - y \subseteq \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i - b_i)$$

Since for all i, I_i is hyperideal, so $a_i - b_i \subseteq I_i$. Thus $x - y \subseteq \sum_{i=1}^{n} I_i$.
Now, let $r \in R$ and $x \in \sum_{i=1}^{n} I_i$, then there exist $a_i \in I_i$ such that $x \in \sum_{i=1}^{n} I_i$Thus $r.x = r.\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} r.a_i$. Since for all i, I_i is hyperideal, then we have $a_i \in I_i$, implies $\sum_{i=1}^{n} r.a_i \subseteq \sum_{i=1}^{n} I_i$. Thus $r.x \in \sum_{i=1}^{n} I_i$. Similarly, we can show that $x, y \in \sum_{i=1}^{n} I_i$. Hence $\sum_{i=1}^{n} I_i$is a hyperideal of R.
Let $x, y \in \prod_{i=1}^{n} I_i$, then for all i there exist $a_{ij}, b_{is} \in I_i$ such that $x \in \sum_{j=1}^{m} a_{1j} a_{2j} \dots a_{nj}$, $y \in \sum_{s=1}^{p} b_{1s} b_{2s} \dots b_{ns}$. Thus

$$x - y \subseteq \sum_{j=1}^{m} a_{1j} a_{2j} \dots a_{nj} - \sum_{s=1}^{p} b_{1s} b_{2s} \dots b_{ns} \subseteq \prod_{i=1}^{n} I_i$$

Let $r \in R$ then

$$r.x = r.\sum_{j=1}^{m} a_{1j} a_{2j} \dots a_{nj} = \sum_{j=1}^{m} r.(a_{1j} a_{2j} \dots a_{nj}) = \sum_{j=1}^{m} (r.a_{1j}) a_{2j} \dots a_{nj}$$

Since I_1 is a hyperideal of R, so so for all j, $ra_{1j} \in I_i$, imples $\sum_{j=1}^{m}(r.a_{1j})a_{2j} \dots a_{nj} \subseteq \prod_{i=1}^{n} I_i$. So $r.x \subseteq \prod_{i=1}^{n} I_i$. On the other hand, since I_n is a hyperideal of R, thus similarly we can show that $x.r \in \prod_{i=1}^{n} I_i$. Hence $\prod_{i=1}^{n} I_i$ is a hyperideal of R.

Theorem 4.5.If I, J and K be hyperideals of hyperring of R. Then,

(i) $I + (J + K) = (I + J) + K$.

(ii) $(IJ)K = I(JK)$.

Proof. (i) Straight forward.

(ii) Let $x \in (IJ)K$, then for all $1 \leq i \leq n$, there exist $a_i, b_i \in I_i$ such that, $x \in \sum_{i=1}^{n} a_i b_i$. Since $a_i \in IJ$ So, for each a_i there exist c_{ij} and d_{ij} such that $a_i \in \sum_{j=1}^{m_i} c_{ij} d_{ij}$ and for all i, j, $c_{ij} \in I, d_{ij} \in J$.

By using associativity and distributivity of R, we have

$$x \in \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m_i} c_{ij} d_{ij}\right) b_i = \sum_{i=1}^{n} \sum_{j=1}^{m_i} c_{ij}\left(d_{ij} b_i\right)$$

Since for all $i, \sum_{i=1}^{n} \sum_{j=1}^{m_i} c_{ij}\left(d_{ij} b_i\right) \subseteq I(JK)$ and $I(JK)$ is a hyperideals of R. Then $x \in I(JK)$. Hence $(IJ)K \subseteq I(JK)$. Similarly, we can show that, $I(JK) \subseteq (IJ)K$. Thus, $(IJ)K = I(JK)$.

Definition 4.6. The hyperideal I is said to be the direct sum of its hyperideals J and K, denoted by $I = J \oplus K$ if for every element $x \in I$ there exist unique elements a, b such that $a \in J, b \in K$ and $x \in a + b$.

Example 4.7.Let $R = \{\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} : m, n \in \mathbb{Z}\}$. Then R with operations defined example of 3.4 is a commutative hyperring.

If $I = \left\{\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} : m \in \mathbb{Z}\right\}$ and $J = \{\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} : n \in \mathbb{Z}\}$. Then I and J are hyperideals of R and $R = I \oplus J$.

Theorem 4.8. Let I, J, K be hyperideals of hyperring of R . Then $I = J \oplus K$ if and only if $I = J + K$and $J \cap K = \{0\}$.

Proof.Suppose $I = J \oplus K$. Then any $x \in I$ can be uniquely written in form $x \in a + b$, where $a \in I$ and $b \in K$. Thuse, in particular $I = J + K$. Now suppose $x \in J \cap K$. Then, $x \in x + 0$ such that $x \in J$, $0 \in K$ and $x \in 0 + x$ such that $0 \in J$, $x \in K$. Since such a sum for x must be unique, so $x = 0$.Thus $J \cap K = \{0\}$.

On the other hand suppose $I = J + K$and $J \cap K = \{0\}$. Show that $I = J + K$. Let $x \in I$, then there exist $a \in J$ and $b \in K$such that $x \in a + b$. Suppose also that $x \in a' + b'$ where $a' \in J$ and $b' \in K$. We need to show that a sum is unique. Since $x \in a + b, x \in á + b́$ and $(I, +)$ is reversible thus $a \in x + (-b)$ and $á \in x + (-b́)$. Hence

$$a - á \subseteq [x + (-x)] + [b́ + (-b)] = 0 + \left(b́ + (-b)\right) = b́ + (-b) = b́ - b$$

Since $b́ - b \subseteq K$, $a - á \subseteq J$, so $a - á \subseteq K$. Thus $a - á = a + (-á) \subseteq J \cap K = \{0\}$. Because the inverse of a is unique thus $a = á$. Similarly it is proved that $b - b́ \subseteq J \cap K = \{0\}$ andso $b = b́$. Thus such a sum for $x \in I$ is unique and so $I = J \oplus K$

Theorem 4.9.Let $I_1, I_2, \dots I_n$ I_n be hyperideals of hyperring R. Then following are equivalent:

(i) $I = \sum_{i=1}^{n} I_i$is direct sum.

(ii $\sum_{i=1}^{n} a_i \in \sum_{i=1}^{n} I_i$ and $\sum_{i=1}^{n} a_i = 0$, then for each i, $a_i = 0$

(iii) $I_i \cap \sum_{j=1(j\neq i)}^{n} I_j = \{0\}$.

Proof. Suppose $\sum_{i=1}^{n} a_i \in \sum_{i=1}^{n} I_i$ and $\sum_{i=1}^{n} a_i = 0$. Since $0 \in 0 + 0 + \dots + 0$ and such a sum for 0 is unique, thus for each i, $a_i = 0$. In other word, (i) implies (ii).

Now let (ii) holds and $\in I_i \cap \sum_{j=1(j\neq i)}^{n} I_j = \{0\}$. Then $x \in I_i$ and $x \in \sum_{j=1(j\neq i)}^{n} I_j$. Since I_i and $\sum_{j=1(j\neq i)}^{n} I_j$ are hyperideals, thus $-x \in I_i$ and $-x \in \sum_{j=1(j\neq i)}^{n} I_j$. Hence $0 = x + (-x) \subseteq I_i \cap \sum_{j=1(j\neq i)}^{n} I_j = \{0\}$ and so $x = 0$.

Thus (ii) implies (iii).

Now if (iii) holds and $x \in \sum_{i=1}^{n} I_i$. Then there exist $a_i \in I_i$ such that $\in \sum_{i=1}^{n} a_i$. Suppose also that $x \in \sum_{i=1}^{n} b_i$, where for each i, $b_i \in I_i$. We show that such a sum for x is unique. The same argument in Theorem 4.5 can be shown that, for each i,

$a_i - b_i \subseteq (a_1 - b_1) + \dots + (a_{i-1} - b_{i-1}) + (a_{i+1} - b_{i+1}) + \dots + (a_n - b_n) \subseteq \sum_{j=1(j\neq i)}^{n} I_j$.

Since $a_i - b_i \subseteq I_i$ for all i, thus $a_i + (-b_i) \subseteq I_i \cap \sum_{j=1(j\neq i)}^{n} I_j = \{0\}$. Hence $a_i + (-b_i) = \{0\}$. Because the inverse of a_i is unique for each i, thus $a_i = b_i$. Hence such a sum for $x \in I$ is unique, and so $I = \sum_{i=1}^{n} I_i$ is direct sum.

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