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Differential Equation of Eigenvalues for Sturm Liouville Boundary Value Problem with Neumann Boundary Conditions

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ABSTRACT: Eigenvalues differential equation of Sturm Liouville boundary value problem is regarded as function of boundary points of a first order differential equation which is true for each regular self-adjoint boundary condition. We study Sturm Liouville differential equation $-(py')'+qy=\lambda\omega y$ Ly=(1)Where $p,q,w: I=(A,B)\to R$ and $-\infty\le A<a<b<B\le+\infty$ $\lambda\in R$ ، $w>0$, $\frac{1}{p},q,w\in L_{loc}(I)$ $J=[a,b]$ on I and we indicate that Neumann's eigenvalues are true in a boundary point in following differential equation: $\lambda'(b)=(q-\lambda w)(b)u^2(b,b)$

***Keywords**:* Eigenvalues, Neumann boundary conditions

INTRODUCTION

When Sturm Liouville differential equation with boundary condition in points a,b with distance (a- b) is considered, boundary value problem is created. Boundary conditions can be stated as general conditions, separate conditions, interrelated conditions, Dirichlet conditions and Neumann conditions. We just introduce separate boundary conditions and its parametric state and Neumann conditions, then necessary propositions and definitions are presented. Regular self-adjoint boundary value problem is introduced and normal function,half-linear Lagrange form which is important in approving propositions, is introduced. It will be shown that Eigenvalues are continuous and there is a normal function $u_n(t,b)$ according to $\lambda_n(b)$ for every $b\in(a,B)$ and $u_n(t,b)$ and $pu'_n(t,b)$ in b are uniform on each compressed interval of (a, B). Then Eigenvalues differential equation of Sturm Liouville problem with Neumann boundary conditions is obtained.

Boundary Conditions

Separated boundary conditions is defined as follows (Kong, 1994):

$$\begin{cases} A_1 y(a)+A_2 py'(a)=0 & \mathrm{A}_1,\mathrm{A}_2\in R \quad A_1^2+A_2^2\neq 0 \\ B_1 y(b)+B_2 py'(b)=0 & \mathrm{B}_1,\mathrm{B}_2\in R \quad \mathrm{B}_1^2+\mathrm{B}_2^2\neq 0 \end{cases}$$

Parametric state of separated boundary condition is defined as follows:

$$\begin{cases} \cos\alpha\, y(a) - \sin\alpha\, py'(a) = 0 & 0 \le \alpha < \pi \\ \cos\beta\, y(b) - \sin\beta\, py'(b) = 0 & 0 < \beta \le \pi \end{cases}$$

If in separated boundary conditions $A_1 = B_1 = 0$, the boundary condition is called Neumann boundary conditions.

Definitions and Propositions

Sturm Liouville problem with general boundary conditions is known as self-adjoint, if:

$$\forall u, v \in D \qquad (\mathrm{Lu,v}) = (\mathrm{u,Lv})$$

Assuming interval (a, b) and coefficients {p,q,w} are given, point a is called regular, if:

$a \in R$ that is $-\infty < a$ b . $P^{-1}, q, \omega \in L^1(a,c] \quad c \in (a,b)$

Sturm Liouville boundary value problem on (a, b) is called regular if a, b are regular.

Complex function f defined on closed interval $I = [a,b]$ is called obviously continuous on I if for each $\varepsilon > 0$ there is a $\delta > 0$, for every nin open intervals: $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ in I, where $\sum_{i=1}^{n} |\beta_i - \alpha_i| < \delta$, then:

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \varepsilon$$

Assuming f is a real function so that $f \in L_{loc}(a,b)$, then:

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f ds = f \text{ Almost everywhere on(a,b)}$$

Theorem: assume $a \in (A,B)$ as fixed, and $F(t) = \int_a^t f(s) ds$. It is approved $F \in AC_{loc}(A,B)$, that is, F is obviously continuous locally on (A, B).

A distinct collection from open intervals $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ is considered:

$$|F(\beta_i) - F(\alpha_i)| = \left| \int_a^{\beta_i} f(s) ds - \int_a^{\alpha_i} f(s) ds \right| \le \int_{\alpha_i}^{\beta_i} |f(s)| ds = |f(s)| (\beta_i - \alpha_i)$$

$$\Rightarrow \sum_{i=1}^{n} |F(\beta_i) - F(\alpha_i)| \le |f(s)| \sum_{i=1}^{n} (\beta_i - \alpha_i) \le \delta |f| = \varepsilon$$

Then $F \in AC_{loc}(A,B)$, thus with $F' \in L^1$ and $F'(t) = f(t)$, now:

$$\frac{1}{h} \int_t^{t+h} f(s) ds = \frac{1}{h} \int_a^{t+h} f(s) ds - \int_a^t f(s) ds = \frac{1}{h} \left(F(t+h) - F(t) \right)$$

$$\Rightarrow \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f(s) ds = \lim \frac{1}{h} [F(t+h) - F(t)] = F'(t) = f(t)$$

And the proposition is approved.

Normal specific function (Kong, 1994):

Special function u is equivalent to Eigenvalue λ in interval $[a,b]$ which is called normal function for Sturm-Liouville equation, if:

$$\int_a^b u^2 w = 1$$

half-linear Lagrange form (Kong, 1994):
When v, u is answers of Sturm- Liouville equation half-linear Lagrange form for v, u is defined as follows:

$$[u,v] = pu\bar{v}' - pu'\bar{v}$$

Assume v, u are answers of Sturm- Liouville equation $Ly = -(py')' + qy$ with $\lambda = \mu$ and $\lambda = \eta$ (Kong, 1994), then:
Theorem: since v, u is an answer of the equation:

$$(pu')' + qu = \mu u w \xrightarrow{\times \bar{v}} \bar{v}(pu')' + qu\bar{v} = \mu u \bar{v} w$$

$$(p\bar{v}')' + q\bar{v} = \eta \bar{v} w \xrightarrow{\times u} u(p\bar{v}')' + qu\bar{v} = \eta u \bar{v} w$$

$$\bar{v}(pu')' + qu\bar{v} - u(p\bar{v}')' - qu\bar{v} = \mu u \bar{v} w - \eta u \bar{v} w$$

$$\Rightarrow \bar{v}(pu')' - u(p\bar{v}')' = (\mu - \eta) u \bar{v} w$$

But:

$$[\bar{v}(pu') - u(p\bar{v}')]' = -u' p\bar{v}' - p(\bar{v}')' u + \bar{v}'(pu') + (pu')' \bar{v}$$

$$= (pu')' \bar{v} - (p\bar{v}')' u$$

$$\Rightarrow \int_a^b [\bar{v} pu' - up\bar{v}']' ds = \int_a^b ((pu')' \bar{v} - (p\bar{v}')' u) ds = \int_a^b (\mu - \eta) u \bar{v} w ds$$

$$\Rightarrow [\bar{v} pu' - up\bar{v}']_a^b = (\mu - \eta) \int_a^b u \bar{v} w ds$$

Continuity of Eigenvalues

When boundary conditions are separate and functions w and $\frac{1}{p}$ and q are not defined on $[a,b]$, $\overline{\Omega}$ is defined as follows:

$$\overline{\Omega} = \left\{ \overline{W} \middle| \overline{W} = (a, b, C, D, \frac{\tilde{1}}{p}, \tilde{q}, \tilde{w}) \right\}$$

Where $\tilde{q} = \begin{cases} q & \text{on } [a,b] \\ 0 & \end{cases}$, $\frac{1}{\tilde{p}}$ and $\tilde{w}$ are defined similarly.
Otherwise,
For space $X = R \times R \times M_{2,2}(C) \times M_{2,2}(C) \times L^1(a,b) \times L^1(a,b) \times L^1(a,b)$ where $-\infty < A < a < b < B < \infty$, norm W is defined as follows:

$$\|W\| = \|\overline{W}\| = |a| + |b| + \|A\| + \|B\| + \int_a^b \left(\left|\frac{1}{p}\right| + |\tilde{q}| + |\tilde{w}| \right) ds$$

$\|A\|$: norm is fixed in each matrix

Assume $W_\circ = (a_\circ, b_\circ, C_\circ, D_\circ, \frac{1}{p_\circ}, q_\circ, w_\circ)$ and $W_\circ \in \Omega$. Assuming $\lambda = \lambda_n(W_\circ)$, n^{th} Eigenvalue of Sturm- Liouville equation is with general boundary condition in $W_\circ$, then λ is continuous in $W_\circ$, that is, for each given $\varepsilon > 0$ there is one $\delta > 0$, so that if $W \in \Omega$ and

$$\|W - W_\circ\| = |a - a_\circ| + |b - b_\circ| + \|C - C_\circ\| + \|D - D_\circ\| + \int_a^b \left(\left|\frac{1}{p} - \frac{1}{p_\circ}\right| + |\tilde{q} - \tilde{q}_\circ| + |w - w_0| \right) < \delta$$

$$\Rightarrow |\lambda(W) - \lambda(W_\circ)| < \varepsilon$$

Uniform convergence theorem (Kong, 1994)

Normal function $u_n(t,b)$ according to $\lambda_n(b)$ is for each $b \in (a, B)$ where $u_n(t,b)$ and $pu'_n(t,b)$ are convergent and uniform on b on each subinterval from (a, B), that is:

$$\lim_{h \to 0} u_n(t, b+h) \to u_n(t,b) \qquad \lim_{h \to 0} pu'_n(t, b+h) \to pu'_n(t,b)$$

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Theorem: assume for each y answer from equation and special functions u(t, b), there is:

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix} \qquad U = \begin{pmatrix} u \\ pu' \end{pmatrix}$$

Assume separate boundary conditions is true, for each h sufficiently small, special function $u = u_n(t, b+h)$ is selected, then uniform convergence $u(t, b+h) \to u(t,b)$ is true on compressed subinterval from continuity λ_n toward b, and answer of y and py' are true on parameter λ, since if:

$$f_m(b) = u_n(t, b + \frac{1}{m}) \quad , \quad f(b) = u_n(t,b) \qquad \forall t \in (a, b)$$

From u_n continuity:

$$\forall \varepsilon > 0 \qquad |f_m(b) - f(b)| = \left| u_n(t, b + \frac{1}{m}) - u_n(t,b) \right| < \frac{\varepsilon}{2}$$

$$\Rightarrow \lim_{m \to +\infty} f_m(b) = f(b)$$

Now:

$$\forall \varepsilon > 0 \qquad \exists N : \forall m_1, m_2 \geq N \qquad \left| f_{m_1}(b) - f_{m_2}(b) \right| = \left| u_n(t, b + \frac{1}{m_1}) - u_n(t, b + \frac{1}{m_2}) \right|$$

$$= \left| u_n(t, b + \frac{1}{m_1}) - u_n(t,b) + u_n(t,b) + u_n(t, b + \frac{1}{m_2}) \right| < \left| u_n(t, b + \frac{1}{m_1}) - u_n(t,b) \right| +$$

$$\left| u_n(t, b + \frac{1}{m_2}) - u_n(t,b) \right| < \varepsilon$$

That is, it was approved $u_n(t,b+h) \to u_n(t,b)$ is uniformly convergent. Regarding $pu'_n(t,b+h)$ it can be written:

$$(pu_n'(t,b))' + qu_n(t,b) = \lambda_n(b)u_n(t,b)w$$

$$(pu_n'(t,b+h))' + qu_n(t,b+h) = \lambda_n(b+h)u_n(t,b+h)w$$

Considering continuity of u_n and λ_n :

$$pu_n'(t,b+h) \to pu_n'(t,b)$$

Eigenvalues Differential equation of Sturm Liouville with Neumann boundary conditions (Anton, 1997)

Sturm Liouville boundary value problem with boundary conditions (3) where $0 \leq \alpha < \pi$ and $\beta = \frac{\pi}{2}$ is considered and it is assumed that $\lambda = \lambda_n$ and $u = u_n$, and:

$$\lambda'(b) = u^2(b,b)(q - \lambda w)(b) \quad \text{Almost everywhere (a,B)}$$

Especially if q, w is continues in b, above relationship is true.

Theorem: for small values of H: $\eta = \lambda(b+h), \mu = \lambda(b)$, $v = u(t,b+h), u = u(t,b)$, then [u,v](a)=0

Because:

If $0 < \alpha < \pi$

$$\cos\alpha\, u(a,b) - \sin\alpha (pu')(a,b) = 0 \Rightarrow pu'(a,b) = \cot g\alpha\, u(a,b)$$

$$\cos\alpha\, u(a,b+h) - \sin\alpha\, pu'(a,b+h) = 0 \Rightarrow pu'(a,b+h) = \cot g\alpha\, u(a,b+h)$$

$$\Rightarrow [u,v](a) = pu(a,b)u'(a,b+h) - pu'(a,b)u(a,ub+h) = u(a,b)\cot g\alpha \times$$

$$u(a,b+h) - \cot g\alpha\, u(a,b)u(a,b+h) = 0$$

If $\alpha = 0$, then u(a,b)=0,u(a,b+h)=0, thus [u,v](a)=

In addition, with $\beta = \frac{\pi}{2}$ and condition $\cos\beta\, u(b,b) - \sin\beta\, pu'(b,b) = 0$:

$$pu'(b,b) = 0 \qquad \forall b \in (a,B) \qquad -\infty \leq A < a < b < B \leq +\infty$$

Considering formulae $[u,v](b)-[u,v](a)=(\mu-\eta)\int_a^b u\bar{v}w(s)ds$:

$$(pu\bar{v}\,')(b)=(\mu-\eta)\int_a^b u\bar{v}w(s)ds$$

$$\Rightarrow -pu(b,b)u'(b,b+h)=(\lambda(b)-\lambda(b+h))\int_a^b u(s,b)u(s,b+h)w(s)ds$$

Since u(s,b) is normal and $u(s,b+h)-u(s,b)\to 0$, by limiting this equality, when $h\to 0$ and by division of above equation by h value:

$$\lambda'(b)=-u(b,b)\lim_{h\to 0}\frac{pu'(b,b+h)}{h}$$

On the other hand:

$$pu'(b,b+h)=(pu')(b,b+h)-(pu')(b+h,b+h)$$

$$=-\int_b^{b+h}(pu')'(s,b+h)ds=-\int_b^{b+h}[q(s)u(s,b+h)-\lambda(b+h)u(s,b+h)w(s)]ds$$

$$=-\int_b^{b+h}[q(s)u(s,b+h)-\lambda(b+h)u(s,b+h)w(s)+q(s)u(s,b)-q(s)u(s,b)+$$

$$\lambda(b+h)u(s,b)w(s)-\lambda(b+h)u(s,b)w(s)]ds=-\int_b^{b+h}q(s)u(s,b)ds+$$

$$\int_b^{b+h}q(s)[u(s,b)-u(s,b+h)]ds-\lambda(b+h)\int_b^{b+h}[u(s,b)-u(s,b+h)]w(s)ds+$$

$$\lambda(b+h)\int_b^{b+h}u(s,b)w(s)ds$$

By division od this equality by h, limiting both sides when $h\to 0$, considering $u(s,b+h)\to u(s,b)$ and $\lambda(b+h)\to\lambda(b)$ and given previous facts, there is:

$$\lim_{h\to 0}\frac{-1}{h}\int_b^{b+h}(q(s)-\lambda(b)w(s))u(s,b)ds=-(q-\lambda w)(b)u(b,b)$$

Thus:

$$\lim_{h\to 0}\frac{(\mathrm{pu'})(\mathrm{b},\mathrm{b}+\mathrm{h})}{h}=-(q-\lambda w)(b)u(b,b)$$

And the proposition is approved by above relationships:

$$\Rightarrow \lambda'(b)=u^2(b,b)(q-\lambda w)(b)$$

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