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Some result about G_2-manifolds and its application in soliton equation

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ABSTRACT: we are going to show that on a compact 7-dimensional manifold which admits a G_2- structure soliton solutions to the Laplacian ow of R. Bryant can only be shrinking or steady. We also show that the space of symmetries (vector fields that annihilate via the Lie derivative) of a torsion-free G_2-structure on a compact 7-manifold is canonically isomorphic to $H^1(M,R)$. Some comparisons with Ricci solitons are also discussed, along with some future directions of exploration.
Keywords: G_2-manifolds, soliton ,symmetry

1 INTRODUCTION

Let M be a 7-dimensional manifold that admits a G_2-structure given by a non-degenerate 3-form ' .A natural geometric flow when M is compact is the Laplacian flow first suggested by R. Bryant in: $\frac{\partial\varphi}{\partial t} = -\Delta_{\varphi}\varphi$
for a family $\varphi = \varphi(t)$ of G_2 -structure, where Δ_{φ} denotes the Hodge Laplacian with respect to the metric induced by $\varphi(t)$ The original intention of the equation (1) is to flow φ to a torsion-free G_2 -structure, since φ being torsion-free is equivalent to being harmonic with respect to the metric g_R it induces.
Definition 1.1. let φ be a G_2 -structure and X a vector field on M.We say that (φ, x) is a Laplacian soliton if equation $\rho\varphi + L_x\varphi = - \Delta\varphi$ is satisfied for some constant ρ
Definition 1.2. Consider the differential 3-form
$\varphi_0 = dx_{123} + dx_{145} +dx_{167} + dx_{246} – dx_{257} – dx_{347} – dx_{356}$
in R^7 where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$ The group G_2 can be defined as the subgroup in GL(7,R) that preserves φ_0 From a principal bundle point of view, a G_2-structure on a 7-dimensional manifold M is just a sub-bundle with structure group G_2, of the GL(7,R) frame bundle over M.
Definition 1.3. We say that a G_2-structure is torsion-free if φ solves the nonlinear system of partial differential equations $\Delta_\varphi= 0$ where ∇ is the covariant derivative induced by g_φ It was shown that a G_2-structure is torsion free if and only if it is closed and co-closed.(with respect to the hodge star induced by g_φ).
Definition 1.4. A 7-manifold M that admits a torsion-free G_2-structure has its Riemannian holonomy (with respect to g_φ) a subgroup of G_2, and such manifolds are simply known as G_2 -manifolds.
Proposition 1.5. Let M be a compact 7-manifold. For any G_2 -structure φ on M, vector field X, and f $\in C^\infty$ (M), we have: $\int_M LX\varphi\wedge *f\varphi = -3\int_M df\wedge *ix\varphi$
proof: We have LX φ = iX dφ+ d iX φ.From the decomposition of d φ we see that
ix dφ $\wedge$* fφ = τ_0 f(ix * φ) $\wedge$*φ + 3f(ix($\tau_1\wedge\varphi$)) $\wedge$*φ
=3f(ix($\tau_1\wedge\varphi$)) $\wedge$*φ + f(ix * τ_3) $\wedge$*φ
= -3f ($\tau_1\wedge\varphi$) $\wedge$(ix*φ) – f *τ_3 $\wedge$ ix *φ
= -3f $\tau_1\wedge$ φ $\wedge$(ix*φ)
= -3f $\tau_1\wedge$ (-4*ixφ)
= 12f $\tau_1\wedge$* ixφ (1)
Where we have used the identity φ $\wedge$(ix*φ) = -4 * ix Δ in the fifth equality and also the point-wise orthogonality of the G_2 -decomposition of differential forms in the second and fourth equalities R above. On the other hand, from the decomposition of d *φ we have

$$\int_M d(ix\ \varphi)\wedge f\ \varphi = \int_M^M ix\ \varphi\wedge *\delta\ f\varphi$$

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$$= -\int_M ix\,\varphi \wedge d * f\,\varphi$$

$$= -\int_M ix\,\varphi \wedge (df \wedge * \varphi + fd * \varphi$$

$$= -\int_M ix\,\varphi \wedge df \wedge * \varphi + \int_M f(\,ix\,\varphi) \wedge (4\tau_1 \wedge * \varphi + * \tau_2)$$

$$= -\int_M df \wedge * \varphi \wedge (ix\varphi) - 4\int_M f(\,ix\,\varphi) \wedge \tau_1 \wedge * \varphi$$

$$= -\int_M df \wedge * \varphi \wedge (ix\varphi) - 4\int_M f\,\tau_1 \wedge * \varphi \wedge (\,ix\,\varphi)$$

$$= -4\int_M df \wedge 3 * ix\varphi - 4\int_M f\,\tau_1 \wedge 3 * ix\,\varphi$$

$$= -3\int_M df \wedge 3 * ix\varphi - 12\int_M f\,\tau_1 \wedge * ix\,\varphi$$

(2) where we have also used the identity $*\varphi \wedge (ix\varphi) = 3 * ix\,\varphi$ Integrating (1) and adding to (2), the lemma now follows.

Lemma 1.6. let φ be a closed G_2-structure. Then φ is an eigen-form if and only if $\Delta_\varphi \in \Omega^3_1$

proof: The only if part is trivial. For the other direction, note that if Δφ = fφ then since φ = 0 we must have d(fφ) = 0 as well. In other words, df∧φ = 0 Recall that the equation above, along with the special form that φ takes with respect to a local orthonormal frame, shows in a straight-forward way that f must be constant. Thus ' must be an eigenform.

Theorem 1.7. If M^7 is compact, then there are no expanding or steady soliton solutions of $-\Delta_d\,\psi + Lx\,\psi = d(ix\,\psi)$ other than the trivial case of a torsion-free G_2-structure in the steady case.

Proof. We take the wedge product of both sides of $\Delta_d\,\psi + Lx\,\psi = d(ix\,\psi)$ with $\psi = d(ix\,\psi)$ $\varphi = *\varphi$ and integrate over M to obtain $\int_M \langle \Delta_d \psi, \psi \rangle\, vol + \lambda \int_M |\psi|^2 vol + \int_M \langle d(ix\,\psi), \psi \rangle vol = 0$

Since M is compact, we have $\int_M \langle d(ix\,\psi), \psi \rangle vol = \int_M \langle ix\,\psi, d * \psi \rangle vol$ But the G_2-structure

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is coclosed,so $\tau_1 = 0$ and henced $*\psi = *d * \psi = *d\varphi = *(\tau_0\varphi + *\tau_3) = \tau_0 + \tau_3$ Therefore $d*\psi$ lies in the space $\Lambda^4_1 \oplus \Lambda^4_{27}$,while iX ψ lies in Λ^4_{27} Since this decomposition of Λ^4 is pointwise orthogonal with respect to the metric g_φ we see that the last term in: $\int_M \langle \Delta_d \psi, \psi \rangle\, vol + \lambda \int_M |\psi|^2 vol + \int_M \langle d(ix\,\psi), \psi \rangle vol = 0$ vanishes. Since $\langle\langle \Delta_d \psi, \psi \rangle\rangle + 7\lambda \int_m vol = ||d * \psi||^2 + 7\lambda vol(M) = 0$ again using the fact that dψ =0.Thus we cannot have λ > 0,and if λ=0 then the G_2-structure must be torsion-free. In the latter case X must be a vector field generating a G_2 -symmetry: LX ψ=0 Since M is compact, there will be no such nonzero X unless M has reducible holonomy.

2 MAIN RESULT

the space of symmetries of ' is isomorphic to $H^1(M,R)$.

Proof. To prove the if part, we will employ the general relation below for the Levi - Civita connection:

$Lx\,\varphi(Y_1, Y_2, Y_3) = \nabla_x\,\varphi(Y_1, Y_2, Y_3) + \varphi(\nabla_{Y_1}X, Y_2, Y_3) + \varphi(Y_1, \nabla_{Y_2}X, Y_3)$, $\varphi(Y_1, Y_2, \nabla_{Y_3}X)$ for any vector fields X, Y_1, Y_2, Y_3. The torsion-free condition is defined by $\nabla_\varphi = 0$. more, since any G_2-manifold must have zero Ricci curvature everywhere, by the Bochner's Theorem we know that any harmonic 1-form must be parallel. Then we must have $\nabla_X = 0$. Combining these facts into above equation we get the desired result.

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