A comparative study of some numerical approaches for mixed integral equations with Adomian method

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ABSTRACT: In this article, numerical solution of Volterra-Fredholm integral equations of mixed type is studied using two classes of the orthogonal functions, two-dimensional block-pulse and two-dimensional triangular functions and their operational matrix. These numerical methods are compared by Adomian's decomposition method. Some numerical examples are provided to illustrate advantages and disadvantages of considered approaches and comparison between them.

Keywords: Two-dimensional Block-pulse functions, Two-dimensional triangular functions, operational matrix, Adomian's decomposition method, Volterra-Fredholm integral equations.

INTRODUCTION

The integral equations method is widely used for solving many problems in science, engineering, physics, and other fields. These problems are often reduced to the Volterra-Fredholm integral equations of mixed type. For example, they are appeared in heat conduction theory and epidemiology (Hacia, 2004 and Diekmann, 1978). The general form of a Volterra-Fredholm integral equations of mixed type (abbreviated as MIEs) can be written as

\[ f(s,t) = g(s,t) + \int_{a}^{t} \int_{\Omega} F(s,t,\xi,\tau, f(\xi,\tau)) d\xi d\tau, \quad (s,t) \in [a,T] \times \Omega, \]

where \( f(s,t) \) is an unknown scalar valued function, \( g(s,t) \) and \( F(s,t,\xi,\tau, f(\xi,\tau)) \) are given functions defined respectively on \([a,T] \times \mathbb{R}\) and \( S \times \mathbb{R} \), where \( S = \{(s,t,\xi,\tau) : 0 \leq \xi \leq s \leq t, t \in \Omega, \tau \in \Omega \} \). The function \( F(s,t,\xi,\tau, f(\xi,\tau)) \) is nonlinear in \( f \). It is assumed that the given functions are such that (1) possesses a unique solution \( f(s,t) \in C([a,T] \times \Omega) \). Existence and uniqueness results for (1) may be found in (Pachpatte, 1986).

Usually, evaluation of the exact solution of integral equations by analytical methods may be difficult, so the numerical methods have a great appeal for researchers. There are a wide range of available methods for numerical solving of these integral equations. The commonly used methods utilize a set of appropriate basis functions and a projection method (collocation, Galerkin) or a direct method. These approaches reduce MIEs to an algebraic equations system. The two-dimensional block-pulse functions (2D-BPFs) and two-dimensional triangular functions (2D-TFs) are sets of orthogonal functions which recently lioness due to their computational powerful properties (Maleknejad et al., 2010, 2011, 2012 and Babolian, 2010). We also apply these functions to solving MIEs. Then we present solution of MIE using Adomian decomposition method that the solution is sum of an infinite series which converges rapidly to the accurate solution. Since, the practical problems in engineering and science require to an appropriate solution with more precision, compared results of 2D-BPFs and 2D-TFs with Adomian method results are considered.

The organization of this paper is as follows. A brief review on 2D-BPFs and 2D-TFs and the direct method for solution MIEs are provided in Section 2 and 3, respectively. Section 4 presents the Adomian decomposition method for MIEs. Section 5 includes some examples to comparison of the methods. Finally, conclusions will be in Section 6.

Numerical solution of MIEs using 2D block-pulse functions

The Block-pulse functions are a set of orthogonal functions with piecewise constant values which was first
introduced to electrical engineers by Harmuth in 1969. Block-pulse functions of one-dimensional have been widely used for solving different problems. A complete details for one-dimensional block-pulse functions is given in (Rao,1983 and Jiang,1992). These discussions can also be extended to the two-dimensional block-pulse functions.

Definitions and properties

Definition 1 In an \((N_1 \times N_2)\)-set of two-dimensional block-pulse functions (2D-BPFs) on the region \((0,1) \times (0,1)\) each component is defined as

\[
\phi_{n_1,n_2}(s,t) = \begin{cases} 
1, & \frac{n_1-1}{N_1} \leq s < \frac{n_1}{N_1}, \quad \frac{n_2-1}{N_2} \leq t < \frac{n_2}{N_2}, \\
0, & \text{Otherwise},
\end{cases}
\]

where \(n_1 = 1,2,\cdots,N_1\) and \(n_2 = 1,2,\cdots,N_2\), for arbitrary positive integers \(N_1\) and \(N_2\).

The 2D-BPFs set has several important properties as follows

- Orthogonality
  \[
  \int_0^{T_1} \int_0^{T_2} \phi_{n_1',n_2'}(s,t) \cdot \phi_{n_1,n_2}(s,t) \, ds \, dt = \begin{cases} 
  h_1 h_2, & n_1 = m_1, n_2 = m_2, \\
  0, & \text{Otherwise}
\end{cases}
  \]

- Completeness
  For every \(f \in L^2((0,T_1) \times (0,T_2))\) when \(m_1\) and \(m_2\) approach to the infinity, Parseval’s identity holds
  \[
  \left\| \int_0^{T_1} \int_0^{T_2} f(s,t) \, ds \, dt \right\|^2 = \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \left\| \phi_{n_1,n_2}(s,t) \right\|^2
  \]

where \(f_{n_1,n_2} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} f(s,t) \phi_{n_1,n_2}(s,t) \, ds \, dt \).

- Disjointness
  \[
  \phi_{n_1,n_2}(s,t) \cdot \phi_{m_1,m_2}(s,t) = \begin{cases} 
  \phi_{n_1,n_2}(s,t), & n_1 = m_1, n_2 = m_2, \\
  0, & \text{Otherwise}
\end{cases}
  \]

The set of 2D-BPFs may be written as a vector \(\Phi(s,t)\) of dimension \(m_1 m_2\)

\[
\Phi(s,t) = [\phi_{1,1}(s,t), \cdots, \phi_{1,m_2}(s,t), \cdots, \phi_{m_1,1}(s,t), \cdots, \phi_{m_1,m_2}(s,t)]^T
\]

where \((s,t) \in (0,T_1) \times (0,T_2))\). Moreover if \(V \) be an \(m_1 m_2\)-vector and \(\tilde{V} = \text{diag}(V)\), then

\[
\Phi(s,t) \Phi^T(s,t) V = \tilde{V} \Phi(s,t)
\]

Moreover, it can be clearly concluded that for every \(m_1 m_2\)-matrix \(A\),

\[
\Phi^T(s,t) A \Phi(s,t) = \hat{A} \Phi(s,t)
\]

in which \(\hat{A}\) is an \(m_1 m_2\)-matrix with elements equal to the diagonal entries of matrix A.

A function \(f(s,t) \in (0,T_1) \times (0,T_2))\), can be expanded by 2D-BPFs as

\[
f(s,t) \equiv \sum_{n_1=1}^{m_1} \sum_{n_2=1}^{m_2} f_{n_1,n_2} \phi_{n_1,n_2}(s,t) = F^T \Phi(s,t) = \Phi^T(s,t) F,
\]

where \(F\) is an \((m_1 m_2)\)-vector given by \(F = [f_{1,1}, \cdots, f_{1,m_2}, \cdots, f_{m_1,1}, \cdots, f_{m_1,m_2}]^T\).

The block-pulse coefficients, \(f_{n_1,n_2}\), are obtained as

\[
f_{n_1,n_2} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} f(s,t) \phi_{n_1,n_2}(s,t) \, ds \, dt.
\]
such that the mean square error between $f(s,t)$, and its block-pulse expansion (5) in the region of $s \in [0,T_1]$ and $t \in [0,T_2)$ is minimal

$$
\varepsilon = \frac{1}{T_1 T_2} \int_{0}^{T_1} \int_{0}^{T_2} \left( f(s,t) - \sum_{n_1}^{m_1} \sum_{n_2}^{m_2} f_{n_1,n_2}(s,t) \right)^2 dsdt.
$$

Similarly a function of four variables, $K(s,t,\xi,\tau)$, on $[0,T_1) \times [0,T_2) \times [0,T_3) \times [0,T_4)$ may be approximated with respect to 2D-BPFs as follows

$$
K(s,t,\xi,\tau) \equiv \Phi^T(s,t)K\Psi'(\xi,\tau),
$$
where $\Phi(s,t)$ and $\Psi(s,t)$ are 2D-BPFs vectors of dimension $m_1m_2$ and $m_3m_4$, respectively, and $K$ is the $(m_1m_2) \times (m_3m_4)$-2D-BPFs coefficients matrix (Maleknejad et al., 2004).

**Solving of MIEs using 2D-BPFs in a direct method**

The results obtained in previous section are used to solve MIEs. Consider the mixed Volterra-Fredholm equation as follows

$$
f(s,t) = g(s,t) + \int_{0}^{s} \int_{0}^{t} k(s,t,\xi,\tau)f(\xi,\tau)d\xi d\tau,
$$

Approximating functions $f$, $g$ and $k$ with respect to 2D-BPFs gives

$$
f(s,t) \equiv F^T \Phi(s,t),
\quad g(s,t) \equiv G^T \Phi(s,t),
\quad K(s,t,\xi,\tau) \equiv \Phi^T(s,t)K\Phi(\xi,\tau).
$$

Substituting (9) into Eq. (8) gives

$$
F^T \Phi(s,t) \equiv G^T \Phi(s,t) + \int_{0}^{s} \int_{0}^{t} F^T(s,t)K\Phi(\xi,\tau)\Phi^T(\xi,\tau)Fd\xi d\tau,
$$

$$
= G^T \Phi(s,t) + \Phi^T(s,t)K \int_{0}^{s} \int_{0}^{t} \Phi(\xi,\tau)\Phi^T(\xi,\tau)Fd\xi d\tau,
$$

$$
\equiv G^T \Phi(s,t) + \Phi^T(s,t)K\tilde{F}Q\Phi(s,t),
$$

in which $K\tilde{F}Q$ is an $(m_1m_2) \times (m_1m_2)$ matrix and $Q$ is the operational matrix of 2D-BPFs as follows

$$
\int_{0}^{s} \int_{0}^{t} \Phi(\xi,\tau)d\xi d\tau = \int_{0}^{s} \Phi(\xi) d\xi \int_{0}^{t} \Phi(\tau) d\tau \equiv Q \cdot \Phi(s,t),
$$

where

$$
Q = \frac{1}{2} \begin{pmatrix}
1 & 1 & 2 & \cdots & 2 \\
0 & 1 & 2 & \cdots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix},
$$

because of $\int_{0}^{t} \Phi(\tau) d\tau = h$ and $\int_{0}^{s} \Phi(\xi) d\xi = P \Phi(s)$, which $P$ is the operational matrix of one-dimensional block-pulse functions. Eq. (4) follows

$$
\Phi^T(s,t)K\tilde{F}Q\Phi(s,t) = \overline{KFQ^T} .\Phi(s,t),
$$

Since $\tilde{F}$ is a diagonal matrix, we have

$$
\overline{KFQ} = \Pi \cdot F,
$$

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in which $\Pi$ is a $(m_1m_2 \times m_1m_2)$-matrix with components

$$\Pi_{i,j} = K_{i,j} \cdot Q_{j,i}, \quad i, j = 1, 2, \ldots, m_1m_2.$$  

Combining and substituting Eqs. (12) and (11) in (10) and replacing $\equiv$ with $=$, follows that

$$F^T \Phi(s, t) = G^T \Phi(s, t) + (\Pi \cdot F)^T \Phi(s, t),$$

or

$$F(I - \Pi) = G,$$  

in which $I$ is $(m_1m_2 \times m_1m_2)$ identical matrix. Eq. (13) is a system of algebraic equations. Components of unknown vector $F$ can be obtained by solving this system using Newton’s or other iterative methods. Hence, an approximate solution $f(s, t) \equiv F^T \Phi(s, t)$ can be computed for Eq. (8).

Two-dimensional triangular functions

One-dimensional triangular functions (1D-TFs) have been introduced by Deb et al. in 2006. The 1D-TFs have been applied for solving different problems. Extending 1D-TFs to two dimensional ones (2D-TFs) was acheived by Babolian et. al.

Definitions and properties

Definition 2 In an m-set of one-dimensional triangular functions (1D-TFs) over interval $[0,1)$, the $i$ th left hand and right hand functions are defined as

$$T_i^1(s) = \begin{cases} 1 - \frac{s - ih}{h}, & \text{if } s < (i+1)h, \\ 0, & \text{otherwise,} \end{cases} \quad T_i^2(s) = \begin{cases} \frac{s - ih}{h}, & \text{if } s < (i+1)h, \\ 1 - \frac{s - ih}{h}, & \text{otherwise,} \end{cases}$$  

where $i = 0, 1, 2, \ldots, m-1$ and $h = 1/m$. The properties of 1D-TFs, were described in (Deb et al., 2006).

Definition 3 An $m_1 \times m_2$-set of 2D-TFs on the region $(0, 1) \times (0, 1)$, are defined as

$$T_{i,j}^{1,1}(s, t) = \begin{cases} (1 - \frac{s - ih_1}{h_1})(1 - \frac{t - jh_2}{h_2}), & \text{if } i_1 \leq s < (i+1)h_1, \quad j_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases}$$

$$T_{i,j}^{1,2}(s, t) = \begin{cases} (1 - \frac{s - ih_1}{h_1})(\frac{t - jh_2}{h_2}), & \text{if } i_1 \leq s < (i+1)h_1, \quad j_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases}$$

$$T_{i,j}^{2,1}(s, t) = \begin{cases} (\frac{s - ih_1}{h_1})(1 - \frac{t - jh_2}{h_2}), & \text{if } i_1 \leq s < (i+1)h_1, \quad j_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases}$$

$$T_{i,j}^{2,2}(s, t) = \begin{cases} (\frac{s - ih_1}{h_1})(\frac{t - jh_2}{h_2}), & \text{if } i_1 \leq s < (i+1)h_1, \quad j_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases}$$

where $i = 0, 1, 2, \ldots, m_1 - 1$, $j = 0, 1, 2, \ldots, m_2 - 1$ and $h_1 = 1/m_1$, $h_2 = 1/m_2$. The values of $m_1$ and $m_2$ are arbitrary positive integers. Note that,

$$T_{i,j}^{1,1}(s, t) + T_{i,j}^{1,2}(s, t) + T_{i,j}^{2,1}(s, t) + T_{i,j}^{2,2}(s, t) = \phi_{i,j}(s, t),$$

where $\phi_{i,j}(s, t)$ is the $(i, j)$ th block-pulse function defined on $ih_1 \leq s < (i+1)h_1$ and $jh_2 \leq t < (j+1)h_2$, as

$$\phi_{i,j}(s, t) = \begin{cases} 1, & \text{if } ih_1 \leq s < (i+1)h_1, \quad jh_2 \leq t < (j+1)h_2, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that
\[ T_{i,j}^{1,1}(s,t) = T_{i}^{1}(s) \cdot T_{j}^{1}(t), \quad T_{i,j}^{1,2}(s,t) = T_{i}^{1}(s) \cdot T_{j}^{2}(t), \]
\[ T_{i,j}^{2,1}(s,t) = T_{i}^{2}(s) \cdot T_{j}^{1}(t), \quad T_{i,j}^{2,2}(s,t) = T_{i}^{2}(s) \cdot T_{j}^{2}(t) \]

in which \( T_{i}^{1}(s) \), \( T_{j}^{1}(t) \), \( T_{i}^{2}(s) \) and \( T_{j}^{2}(t) \) are defined in (14).

Similar to the one-dimensional triangular functions, there are some properties for 2D-TFs as follows:

- **Disjointness.** Each set of \( \{ T_{i,j}^{1,1}(s,t) \}, \{ T_{i,j}^{1,2}(s,t) \}, \{ T_{i,j}^{2,1}(s,t) \} \) and \( \{ T_{i,j}^{2,2}(s,t) \} \) are obviously disjoint, namely

  \[ T_{i,j}^{p,1,q_1}(s,t) \cdot T_{i,j}^{p_2,q_2}(s,t) = \begin{cases} 1, & \text{if } p = p_2, q_1 = q_2, i = i_2, j = j_2, \\ 0, & \text{otherwise}, \end{cases} \]

  for \( p, q \in \{1,2\}, i_1, i_2 = 0, 1, 2, ..., m_{1} - 1, \) and \( j_1, j_2 = 0, 1, 2, ..., m_{2} - 1. \)

- **Orthogonality.** The 2D-TFs are orthogonal, that is

  \[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} T_{i,j}^{p_1,q_1}(s,t) \cdot T_{i',j'}^{p_2,q_2}(s,t) \, ds \, dt = \Delta_{p_1,p_2} \cdot \Sigma_{q_1,q_2} \cdot \delta_{i,j}, \]

  where \( \delta \) denotes Kronecker delta function, and

  \[ \Delta_{\alpha,\beta} = \begin{cases} h/3, & \alpha = \beta \in \{1,2\}, \\ h/6, & \alpha \neq \beta. \end{cases} \]

The 2D-TFs vector, can be defined as

\[ T(s,t) = \begin{bmatrix} T_{11}(s,t) \\ T_{12}(s,t) \\ T_{21}(s,t) \\ T_{22}(s,t) \end{bmatrix} \in \mathbb{R}^{4m_{1}m_{2} \times 1} \]

if

\[ T_{11}(s,t) = \begin{bmatrix} T_{0,0}^{1,1}(s,t) & T_{0,1}^{1,1}(s,t) & \cdots & T_{m_{1},m_{2}-1}^{1,1}(s,t) \end{bmatrix}, \]
\[ T_{12}(s,t) = \begin{bmatrix} T_{0,0}^{1,2}(s,t) & T_{0,1}^{1,2}(s,t) & \cdots & T_{m_{1},m_{2}-1}^{1,2}(s,t) \end{bmatrix}, \]
\[ T_{21}(s,t) = \begin{bmatrix} T_{0,0}^{2,1}(s,t) & T_{0,1}^{2,1}(s,t) & \cdots & T_{m_{1},m_{2}-1}^{2,1}(s,t) \end{bmatrix}, \]
\[ T_{22}(s,t) = \begin{bmatrix} T_{0,0}^{2,2}(s,t) & T_{0,1}^{2,2}(s,t) & \cdots & T_{m_{1},m_{2}-1}^{2,2}(s,t) \end{bmatrix}. \]

A function \( f(s,t) \in ([0,1] \times [0,1]) \) may be extended by 2D-TFs as

\[ f(s,t) = \sum_{i=0}^{m_{1}-1} \sum_{j=0}^{m_{2}-1} c_{i,j} T_{i,j}^{1,1}(s,t) + \sum_{i=0}^{m_{1}-1} \sum_{j=0}^{m_{2}-1} d_{i,j} T_{i,j}^{1,2}(s,t) + \sum_{i=0}^{m_{1}-1} \sum_{j=0}^{m_{2}-1} e_{i,j} T_{i,j}^{2,1}(s,t) + \sum_{i=0}^{m_{1}-1} \sum_{j=0}^{m_{2}-1} f_{i,j} T_{i,j}^{2,2}(s,t) \]

\[ = C^{T} \cdot T_{11}(s,t) + C^{2T} \cdot T_{12}(s,t) + C^{3T} \cdot T_{21}(s,t) + C^{4T} \cdot T_{22}(s,t) = C^{T} \cdot T(s,t), \]

where \( T(s,t) \) is defined in (16) and \( C \) is a \( 4m_{1}m_{2} \)-vector given by

\[ C = \begin{bmatrix} C^{1T} \quad C^{2T} \quad C^{3T} \quad C^{4T} \end{bmatrix}. \]

The coefficients \( C^{1}, C^{2}, C^{3}, \) and \( C^{4} \) can be computed by sampling the function \( f(s,t) \) at grid points \( s_{i} \) and \( t_{j} \) such that \( s_{i} = ih_{1} \) and \( t_{j} = jh_{2} \), for various \( i \) and \( j \). Therefore,
\[ C_{I_k} = c_{i,j} = f(s_i, t_j), \quad C_{2k} = d_{i,j} = f(s_{i+1}, t_j), \]
\[ C_{3k} = e_{i,j} = f(s_{i+1}, t_{j+1}), \quad C_{4k} = l_{i,j} = f(s_{i+1}, t_{j+1}), \]
where \( k = im_2 + j \) and \( i = 0,1,\cdots,m_1-1, j = 0,1,\cdots,m_2-1 \) (Babolian et al., 2010).

Let \( X \) be a \( 4m_1m_2 \)-vector as \( X = \begin{bmatrix} X_{1T} & X_{2T} & X_{3T} & X_{4T} \end{bmatrix} \)
where \( X_1, X_2, X_3 \) and \( X_4 \) are \( m_1m_2 \)-vectors. It can be concluded that
\[ T(s,t) \cdot T^T(s,t) \cdot X = V(t) \cdot \tilde{X} \cdot Ts(s), \] (18)
in which \( Ts(s) \) and \( V(t) \) are the s-components and t-components of \( T(s,t) \cdot T^T(s,t) \) and \( \tilde{X} = \text{diag}(X) \). For more details, see (Maleknejad et al., 2012).

**Solving the MIEs using 2D-TFs in a direct method**

Approximating \( f(s,t), g(s,t) \) and \( k(s,t,\xi,\tau) \) with respect to 2D-TFs, gives
\[ f(s,t) \equiv F^T \cdot T(s,t) = T^T(s,t) \cdot F, \]
\[ g(s,t) \equiv G^T \cdot T(s,t) = T^T(s,t) \cdot G, \]
\[ k(s,t,\xi,\tau) \equiv T^T(s,t) \cdot K \cdot Ts(s)(\xi,\tau), \] (19)
where \( T(s,t) \) is defined in Eq. (16). Substituting Eq. (19) in Eq. (8), obtains
\[ T^T(s,t) \cdot F \equiv T^T(s,t) \cdot G + \int_0^1 T^T(s,t) \cdot K \cdot T(\xi,\tau) \cdot T^T(\xi,\tau) \cdot F d\xi d\tau, \]
\[ T^T(s,t) \cdot G \equiv T^T(s,t) \cdot G + \int_0^1 T^T(s,t) \cdot K \cdot \int_0^1 T(\xi,\tau) \cdot T^T(\xi,\tau) \cdot F d\xi d\tau, \]
\[ = T^T(s,t) \cdot G + T^T(s,t) \cdot K \cdot \int_0^1 V(\tau) \cdot \tilde{F} \cdot Ts(\xi) d\xi d\tau. \] (20)

Approximating of the first integral in Eq. (20) yields
\[ \int_0^1 V(\tau) d\tau \equiv \begin{bmatrix} (h_2/3)I_1 & (h_2/6)I_1 & 0 & 0 \\ (h_2/6)I_1 & (h_2/3)I_1 & 0 & 0 \\ 0 & 0 & (h_2/3)I_1 & (h_2/6)I_1 \\ 0 & 0 & (h_2/6)I_1 & (h_2/3)I_1 \end{bmatrix} = H, \] (21)
where \( I_1 = I_{m_1m_2 \times m_1m_2} \) and approximating of the second integral in Eq. (20) yields
\[ \int_0^1 Ts(\xi) d\xi \equiv W \cdot T(s,t), \] (22)
where \((4m_1m_2 \times 4m_1m_2)\)-matrix \( W \) is as follows
\[ W = \begin{bmatrix} P_1s & P_2s & P_2s & P_2s \\ P_1s & P_2s & P_2s \\ P_1s & P_2s & P_2s \\ P_1s & P_2s & P_2s \end{bmatrix} \otimes I_{m_2 \times m_2}, \]
in which \( P_{1s_{\text{mod}}} \) and \( P_{2s_{\text{mod}}} \) are the operational matrices of integration in 1D-TF domain, can be represented as
For more details, see (Maleknejad et al., 2012). Substituting (21) and (22) in (20), gives
\[ T^T(s, t) \cdot F \equiv T^T(s, t) \cdot G + T^T(s, t) \cdot \left( K \cdot H \cdot \tilde{F} \cdot W \right) \cdot T(s, t) \equiv T^T(s, t) \cdot G + \left( KHFW \right) \cdot T(s, t), \]
where \((KHFW)\) is a \(4m_1m_2\)-vector with components equal to the diagonal components of the matrix \(KHFW\).

Since \(\tilde{F}\) is a diagonal matrix, we get \((KHFW) = \Pi \cdot F\), in which \(\Pi\) is a \((4m_1m_2 \times 4m_1m_2)\)-matrix with components \(\Pi_{i,j} = (KH)_{i,j} \cdot W_{j,i}, \quad i, j = 1, 2, \ldots, 4m_1m_2\).

So, we have
\[ T^T(s, t) \cdot F = T^T(s, t) \cdot G + T^T(s, t) \cdot \Pi \cdot F, \]
or
\[ F(I - \Pi) = G. \quad (23) \]
Eq. (23) is a system of \(4m_1m_2\) algebraic equations. Components of unknown vector \(F\) can be obtained by solving this system. Hence, an approximate solution \(f(s, t) \equiv F^T \cdot T(s, t)\) can be computed for Eq. (8).

**Adomian decomposition method**

Adomian has developed a numerical technique for solving functional equations (Adomian et al., 1985). In this method, special kinds of polynomials, called Adomian polynomials, are used which can be easily derived. The solution is given by a series in which each term is easily obtained. This method has applied to solve various problems. In (Maleknejad et al., 1999), the Adomian decomposition method is applied for mixed nonlinear Volterra-Fredholm integral equations. We present outline of this method to compare it with direct method. The canonical form of Eq. (1) is
\[ f(s, t) = g(s, t) + K(f), \quad (24) \]
in which \(K(f)(s, t) = \int_0^T f(s, t, \xi, \tau, f(\xi, \tau))d\xi d\tau, \)

In Adomian decomposition method, the solution \(f\) is written \(f(s, t) = \sum_{i=0}^{\infty} f_i\) and \(f(s, t) = g(s, t) + \sum_{k=0}^{\infty} A_k f\).

The Adomian polynomials \(A_k\) for (1) is
\[ A_k = \frac{1}{k!} \int_0^T \sum_{n=0}^{\infty} \frac{d^n}{d\tau^n} \left[ F(s, t, \xi, \tau, f_n(\xi, \tau)) \right] d\xi d\tau. \quad (25) \]

Eq. (25) for Adomian’s polynomials is not very practical to calculate. In (Abbaoui et al., 1994), a new formula is proposed for calculating the Adomian polynomials, without supposing the convergence of the series.
\[ A_k(Kf) = \sum_{p_1 + \cdots + p_k = k} f_{p_1}^{p_1} f_{p_2}^{p_2} \cdots f_{p_k}^{p_k} \left( \frac{d^{p_1+p_2+\cdots+p_k}}{df^{p_1+p_2+\cdots+p_k}} K(f) \right)_{f=f_0}. \quad (26) \]

So, the Adomian method for solution (1) is as
\[ f_0(s, t) = g(s, t) \]
\[ f_{k+1}(s, t) = \int_0^T A_k(F(s, t, \xi, \tau, f(\xi, \tau))d\xi d\tau, \]
where
\[ A_k( F(s,t,\xi,\tau, f(\xi, \tau)) = \sum_{p_1+2p_2+\cdots+kp_k = k} \frac{f_1^{p_1} f_2^{p_2} \cdots f_k^{p_k}}{p_1! p_2! \cdots p_k!} F^{(p_1+p_2+\cdots+p_k)}(s,t,\xi,\tau, f_0(\xi, \tau)). \] (27)

The convergence theorems of Adomian decomposition scheme may be found in (Gabet, 1994).

**Numerical Examples**

In this section, Adomian’s decomposition method which has been stated in (Maleknejad et al., 1999) is compared with direct method using 2D-BPFs and 2D-TFs by some examples. One of examples is considered nonlinear to state efficiency all of stated methods for these equations. It should be mentioned that in direct methods, integral equations are converted to the nonlinear algebraic equations system (Maleknejad et al., 2012). All computations were carried out using Matlab7.

Example 1. Consider the following mixed integral equation,

\[ f(s,t) = g(x,t) - \int_0^1 \int_0^1 \cos(s-\xi) \exp(-(t-\tau)) f(\xi, \tau) d\xi d\tau, \]

where \( g(s,t) = e^{-t}(\cos s + t \cos s + t/2 \cos(s - 2) \sin 2) \), with exact solution \( f(s,t) = e^{-t} \cos(s) \). The absolute errors in some arbitrary points are shown in Table 1.

Example 2. Consider the following integral equation,

\[ f(s,t) = g(s,t) + \int_0^1 \int_0^1 \frac{s(1-\xi^2)}{(1+t)(1+\tau^2)} (1-\exp(-f(\xi, \tau))) d\xi d\tau, \]

with \( g(s,t) = -\ln \left( \frac{1+st}{1+t^2} \right) + \frac{st^2}{8(1+t)(1+t^2)} \), with the exact solution \( f(s,t) = -\ln \frac{1+st}{1+t^2} \). The numerical results are shown in table 2.

The results in Tables 1 and 2 confirm the advantageous of the Adomian method to other considered methods viewpoint of accuracy.
CONCLUSION

The effective numerical methods using 2D-BPFs and 2D-TFs were studied for solving Volterra-Fredholm integral equations of mixed type in a direct method. These methods have some advantages. They transform a mixed integral equation to a system of algebraic equations without applying any integration and projection method. So they are simple to apply for mixed integral equations. In order to determination of an approximate solution with higher accuracy, they were compared with Adomian decomposition method. The results showed Adomian method is more precise than the other considered methods but it has more computations than the other methods.

REFERENCES