An Accurate Approximation to 3-Parameter Weibull Sum Distribution

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ABSTRACT: Sum of 3-parameter Weibull is used in communications systems engineering, in reliability engineering and failure analysis. Unfortunately, closed form expressions do not exist for the distribution of 3-parameter Weibull sum. So far no approaches have been proposed for approximation of 3-parameter Weibull sum. An accurate approximation to Rayleigh (2-parameter Weibull with shape parameter equal two) sum is proposed by Jeremiah and Beaulieu, and a simple precise approximation to 2-parameter Weibull sum based on general Gamma distribution is proposed, but this approximation cannot be generalized to a 3-parameter Weibull distribution. In this letter, we present an accurate approximation to a 3-parameter Weibull sum distribution that is based on Generalized Extreme Value (type II) RV.

Key Words: 3-parameter Weibull, Generalized Extreme Value (type II), Monte Carlo simulation.

SUMMARY

The Weibull distribution is used in reliability studies, for example to study the voltage breakage of electric circuits (Cacciari et al., 1996; Hirose, 1996). It may be used to represent manufacturing and delivery times in industrial engineering problems. It is also a very popular statistical model in reliability engineering and failure analysis, while it is widely applied in radar systems to model the dispersion of the received signal level produced by some types of clutters. Furthermore, concerning wireless communications, the Weibull distribution may be used for fading channel modeling. The accurate computation of the cumulative probability distribution function (CDF) of a sum of 3-parameter Weibull is used in wireless communication.

Filho and Yacoub proposed a precise approximation of the 2-parameter Weibull distribution based on General Gamma (3P) distribution (Filho and Yacoub, 2006; Jeremiah and Beaulieu, 2005). At first, it seems with adding a shifted parameter to Weibull, the sum can be approximated with General Gamma (4P). After fitting many distributions to simulated data we conclude the best approximation for Weibull(3P) sum is based on General Extreme Value (type II) RV.

Cdf Approximation

The Weibull distribution with non-zero shift has three parameters, denoted in the following: $\alpha > 0$, the scale parameter; $\beta > 0$, the shape parameter; and $\gamma$, the shift parameter. Let $X_i, \ i = 1, \ldots, n$ be independent, identically distributed (iid) 3-parameter Weibull RVs. Its probability density function is given by

$$f_{X_i}(x) = \beta \alpha^{-\beta} (x - \gamma)^{\beta - 1} \exp\left(-\frac{x - \gamma}{\alpha}\right); \quad x \geq \gamma.$$  (1)

This is the most general form of the classical Weibull distribution. The CDF of $X_i$ is given by

$$F_{X_i}(x) = 1 - \exp\left(-\frac{x - \gamma}{\alpha}\right).$$  (2)
Suppose Y has Generalized Extreme Value (GEV) type. The CDF of Y is given by

\[
F_Y(y) = \exp \left( - \left( 1 + \frac{y - \mu}{\sigma} \right)^{-\frac{1}{\xi}} \right), \quad \xi > 0, 1 + \frac{y - \mu}{\sigma} > 0 \tag{3}
\]

Here \(\mu, \sigma,\) and \(k\) are the location, scale, shape parameters, respectively. Hosking, Wallis, and Wood (1985) have discussed the method of probability-weighted moments (PWM) for the estimation of the parameters \(\mu, \sigma,\) and \(k\) (Hosking et al., 1985). From the PDF of Generalized Extreme Value Y, three moments of Y could be obtain as follow

\[
E(Y) = \mu + \frac{\sigma}{\xi}(g_1 - 1) \quad 0 < \xi < 1 \tag{4}
\]

\[
V(Y) = \frac{\sigma^2}{\xi^2}(g_2 - g_1^2) \quad 0 < \xi < \frac{1}{2} \tag{5}
\]

\[
SK(Y) = \frac{g_3 - 3g_1g_2 + 2g_1^3}{(g_2 - g_1^2)^{3/2}} \tag{6}
\]

Where

\[
g_k = \Gamma(1 - k\xi).
\]

It can be shown that the distribution of \(\Lambda\) is well approximated by the distribution of \(Y\) with appropriate parameters. So, to approximate the 3-parameter Weibull sum, let

\[
\Lambda = \sum_{i=1}^{N} X_i \approx Y
\]

Where \(X_i\)s are Weibull (3P) RVs, Y is a Generalized Extreme Value RV. This approximation using moments matching to compute the three parameters of Y.

**Moments Matching**

We will use Moments Matching to compute the three parameters \(\mu, \sigma,\) and \(\xi:\)

\[
E(\Lambda) = E(Y) \tag{7}
\]

\[
E(\Lambda^2) = E(Y^2)
\]

\[
E(\Lambda^3) = E(Y^3)
\]

The left side parts Eq. (8) could be obtained easily from the Monte Carlo Simulation denoted as:

\[
E(\Lambda) = m_1
\]

\[
E(\Lambda^2) = m_2 \tag{8}
\]

\[
E(\Lambda^3) = m_3
\]

Now we obtain three equations:
\[ m_1 = \mu + \frac{\sigma}{\xi} (g_1 - 1) \]
\[ m_2 = \mu^2 + \frac{2\mu\sigma}{\xi} (g_1 - 1) + \frac{\sigma^2}{\xi^2} (g_2 - 2g_1 + 1) \]  \hspace{1cm} (9)
\[ m_3 = \mu^3 + 3\mu^2 \frac{\sigma}{\xi} (g_1 - 1) + 3\mu \frac{\sigma^2}{\xi^2} (g_2 - 2g_1 + 1) + \frac{\sigma^3}{\xi^3} (g_3 + 3g_1 - 3g_2 - 1) \]

Solving these equations, we have:
\[ \mu = m_1 - \frac{\sigma}{\xi} (g_1 - 1) \]
\[ \sigma = \xi \sqrt{\frac{(m_2 - m_1^2)}{g_2 - g_1^2}} \]  \hspace{1cm} (10)

With setting these two equations in third Eq.9 and using computing softwares for instance MATHEMATICA we can numerically solve for \( \xi \).

**RESULT**

In this Section, sample numerical examples are given to illustrate the excellent performance of the proposed approximations. The cumulative distribution function (CDF) of the sum of \( N \) 3-parameter Weibull RVs generated by Monte Carlo simulation is used as reference.

![Figure 1. The CDF of sum Weibull(3P) for value of N=1,10,15,20 with \( \alpha = 4, \beta = 3 \) and \( \gamma = 2 \) in Weibull(3P) probability plot](image-url)
Fig 1 shows the CDF of sum Weibull(3P) for values of $N=1,10,15,20$ with scale=4, shape=3, location=2 plotted on "3-parameter Weibull probability paper". As this figure shows the sum distribution is not 3-parameter Weibull approximately.
Figure 4. The CDF of a sum of 10 Weibull(3P) with

Figure 5. The CDF of a sum of 10 Weibull(3P) with
We explain the results of the fitting GEV and General Gamma (4P) to the simulated data. For fitting GEV to the data we need estimate parameters. By using Eq.10 we estimate parameters of GEV distribution. Figs 2,3,4,5 show approximation to the sum of Weibull (3P) RVs for some different values of $\alpha, \beta, \gamma$ and compare with the fitting results which are based on the GEV and General Gamma (4P) distributions. Also Figs 2,3,4 show the GEV distribution as well as General Gamma(4P) distribution approximated sum of Weibull(4P) but Fig 5 shows the excellent performance of the GEV rather than General Gamma(4P).

Therefore, we conclude that although General Gamma (4P) has more parameters than GEV, but it is not better than GEV for approximating 3-parameter Weibull sum distributions and GEV is strictly better than General Gamma (4P) in some cases.

**CONCLUSION**

The GEV approximation is proposed to approximate the entire sum of independent 3-parameter Weibull distributions. Numerical results show that is better approximation than General Gamma (4P). Although, it has less parameters.

**REFERENCES**


