International Research Journal of Applied and Basic Sciences
© 2016 Available online at www.irjabs.com
ISSN 2251-838X / Vol, 10 (7): 917-922
Science Explorer Publications

Chebyshev wavelets Solution for Optimal Control

Mohsen Shahrezaee

Department of Mathematics, Faculty of Science, University of Imam Hussein (AS), Tehran, Iran

Corresponding Author email: mshahrezaee@iust.ac.ir

ABSTRACT: In this paper, a numerical method is introduced for solving the optimal control problems. The method is a state parameterization technique which can consider the state parameters as a linear combination of Chebyshev polynomials with unknown coefficients. In the proposed method, the main object is to reduce the system dimensions to achieve a robust and easy solution. This method approximates the control and state variables as a function of time. In addition, by combining the Wavelet and Chebyshev polynomials, the method is transformed into a more successful technique to find the optimal control solution by comparing with the other existing mentioned algorithms.
Keywords: Chebyshev wavelets polynomial; Optimal control; State parameterization.

INTRODUCTION

Optimal control is a subcategory of mathematical optimization which is extended from the calculus of variations theory. In other words, optimal control can be considered as a control strategy in the control theory. The method is largely due to the work of Bellman's dynamic programming [1] and Pontryagin's maximum principle method [2] which are transformed to the most reknowned methods for solving the optimal control problems.

A large number of problems in the mathematical real world can be modeled as the form of optimal control equations.

Since, the study of optimal control problems and analyze the methods for solving them, are so effective in application. In recent years, several accurate and simple techniques are presented based on orthogonal functions to do this job [3]. Some of these techniques are: Walsh series [4], block-pulse functions [5], Laguerre series [6], Chebyshev polynomials [7]. Wavelet polynomial are one of the relatively new approaches which are being employed for solving a wide range of real world problems [8]. There exist a lot of attempts to increase the accuracy of this structure. Wavelet polynomials provide an accurate representation for different variety of operators and functions.

All of these methods are totally based on converting the dynamic optimal control problem into a quadratic programming problem or converting the two-point boundary value problem into the algebraic equations.

Spectral method is another name for the orthogonal methods [9]. Such a naming is due to the reason that they convert the differential equations into an integral equation through integration.

In a considered equation, orthogonal functions approximate the control and/or state by finite terms and employing an operational matrix of integration to eliminate the integral operations.

The main purpose in this paper is to introduce an orthogonal based method for solving optimal control problems via a hybrid method of Chebyshev and wavelet polynomials. Here, The Chebyshev wavelets properties are utilized to convert the optimal control into a linear system of algebraic equations.

In this paper, the algorithm presented in [6] is modified, and an impressive iterative algorithm is proposed. In this approach, there is only one unknown coefficient to find the problem approximation. This property makes the system more robust against the other ordinary orthogonal methods.

Chebyshev Wavelets

Wavelets form a family of functions which are generated from translation and dilation of a described function called the mother wavelet. By continuously varying the dilation (*a*) and the translation (*b*) parameters and assuming a normalized time (*t*), wavelet functions can be considered as below:

$$\Psi_{a,b}(\mathrm{t}) = |a|^{-\frac{1}{2}} \Psi\left(\frac{t-b}{a}\right), \quad a,b \in \mathbb{R}, a \neq 0 \qquad (1)$$

Consider the Chebyshev wavelets as $\Psi(t)=\Psi(k,m,n,t)$ with $k=1,2,\ldots$, $n=1,2,\ldots,2^k$ and *m* as the order of the chebyshev polynomial. By considering the above assumptions, the hybrid chebyshev wavelet polynomial can be described as:

$$\Psi_{nm}(t)=\begin{Bmatrix} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^{k+1}t-2n+1), & \frac{n-1}{2^k}\le t\le \frac{n}{2^k} \\ 0 & o.w. \end{Bmatrix} \qquad (2)$$

and,

$$\alpha_m(t)=\begin{Bmatrix} \sqrt{2}, & m=0 \\ 2 & m=1,2,\ldots \end{Bmatrix} \qquad (3)$$

where, $T_m(t)$ is the chebyshev polynomial of order *m* [6]. The chebyshev polynomial can be formulated by a three-part recursive formulas below [10]:

$$T_0(x)=1, \qquad (4)$$
$$T_1(x)=x,$$
$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x) \quad n=1,2,3,\cdots$$

Therefore, the total Chebyshev wavelet approximation with truncation can be considered as below:

$$f(t)\approx\sum_{n=1}^{2^k}\sum_{m=0}^{m-1} f_{nm}\Psi_{nm}(t)=F^T\Psi \qquad (6)$$

Note that Chebyshev wavelet function is defined in the range $[0,1)$.

Problem statement

The behavior of a nonlinear differential system with the fixed time interval $[t_0,t_1]$ can be considered as below:

$$U(\tau)=f(\tau,X(\tau),\dot{X}(\tau)), \qquad (1)$$

By the following initial conditions:

$$X(t_0)=x^0, \quad X(t_1)=x^1, \qquad (2)$$

Here $X(.)$, $U(.)$ are the state and the control variables and *f* describes a real-valued continuous differentiable function.

The main purpose in the optimal control is to find control $U(.)$, from the initial position $x(t_0)=x_0$ to position $x(t_1)=x_1$ within the time (t_1-t_0) and achieving the optimal performance index, *PI*, which can be illustrated as:

$$PI=\int_{t_0}^{t_1} L(\tau,X(\tau),U(\tau))d\tau \qquad (3)$$

The main purpose of the control in here is to minimize *PI*. If $t_0\neq 0$ or $t_1\neq 1$, then for employing the Chebyshev wavelet polynomials, we introduce the transformation as below:

$$\tau=(t_1-t_0)t+t_0, \qquad (4)$$

After transformation, the optimal control variables, its initial conditions for the trajectory $x(t)$ and the performance index can be written as below:

$$u(t)=f\left((t_1-t_0)t+t_0,x(t),\dot{x}(t)\right), \qquad (5)$$

$$x(0)=x_0, \quad x(1)=x_1 \qquad (6)$$

$$J(x)=(t_1-t_0)\int_{-1}^{1} L\left((t_1-t_0)t+t_0,x(t),u(t)\right)dt \qquad (7)$$

By considering the considered Chebyshev Wavelet basis as $\Psi_{nm}(t)$ and *k*=0, the approximation for $x(\cdot)$ can be achieved as below:

$$x_1(t) = \sum_{i=0}^{2} a_i \Psi_{nm}(\mathrm{t}), \tag{8}$$

Hence, the total approximation solution can be evaluated as:

$$x_{n+1}(t) = x_n(t) + \sum_{i=n}^{n+2} a_i U_i(t), \tag{9}$$

By the boundary conditions as:

$$x_{n+1}(-1) = x_n(-1) = x_0 \tag{10}$$

$$\Rightarrow a_{n+2}U_{n+2}(-1) + a_{n+1}U_{n+1}(-1) + a_n U_n(-1) = 0$$

and

$$x_{n+1}(1) = x_n(1) = x_1 \tag{11}$$

$$\Rightarrow a_{n+2}U_{n+2}(1) + a_{n+1}U_{n+1}(1) + a_n U_n(1) = 0$$

The Numerical Results

To illustrate the approximation accuracy, two continuous optimal control problems are analyzed and the achieved results are compared with the exact solution. All of the considered problems have analytical solution which let us to characterize and to validate the proposed method by comparison with results of exact solutions.

Example 1

Consider the differential equation of the form:

$$J = \int_0^1 \left(U(\tau)^2 + X(\tau)^2 \right) d\tau, \quad 0 \le \tau \le 1 \tag{12}$$

The main object is to find the optimal state and optimal control values by minimizing the performance index and subject to:

$$U(\tau) = \dot{X}(\tau), \tag{13}$$

The boundary conditions are:

$$X(0) = 0, \tag{14}$$

$$X(1) = \frac{1}{2}, \tag{15}$$

The exact solution for this problem can be evaluated by Pontryagin's maximum principle method as below [11]:

$$X(\tau) = \frac{e(e^{\tau} - e^{-\tau})}{2(e^2 - 1)}, \tag{16}$$

$$U(\tau) = \frac{e(e^{\tau} + e^{-\tau})}{2(e^2 - 1)}, \tag{17}$$

The optimal value for the index performance (*PI*) which is calculated by the presented method is illustrated in the Table below. The results for the proposed method are shown in figures 1 and 2.

Table 1. Estimated and exact values of *PI* for Example 1.

Iteration	Proposed Method	Error	Mehne Method [16]	Error	Kafash Method [1]	Error
1	0.3286	$3.1e^{-4}$	0.333	$5.0e^{-3}$	0.3285	$3.3e^{-4}$
2	0.3282	$4.9.0\ e^{-7}$	0.3285	$3.4\ e^{-3}$	0.3282	$5.2e^{-7}$
3	0.3282	$8.87\ e^{-9}$	0.3284	$2.1\ e^{-4}$	0.3282	$1.6e^{-8}$

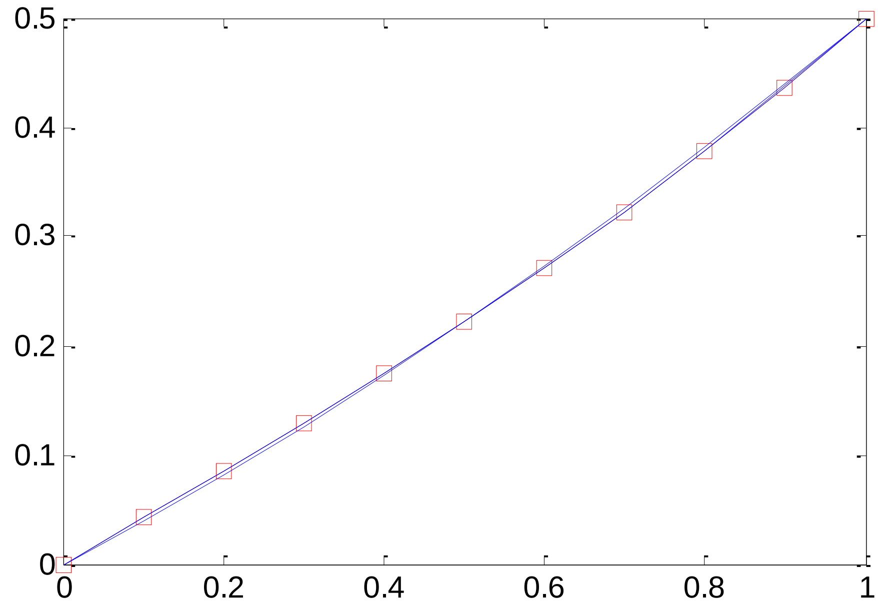

Figure 1. Solution of Example 1: state approximate solution for n=1,2,3 (blue lines) are compared with the actual analytical solution (red square)

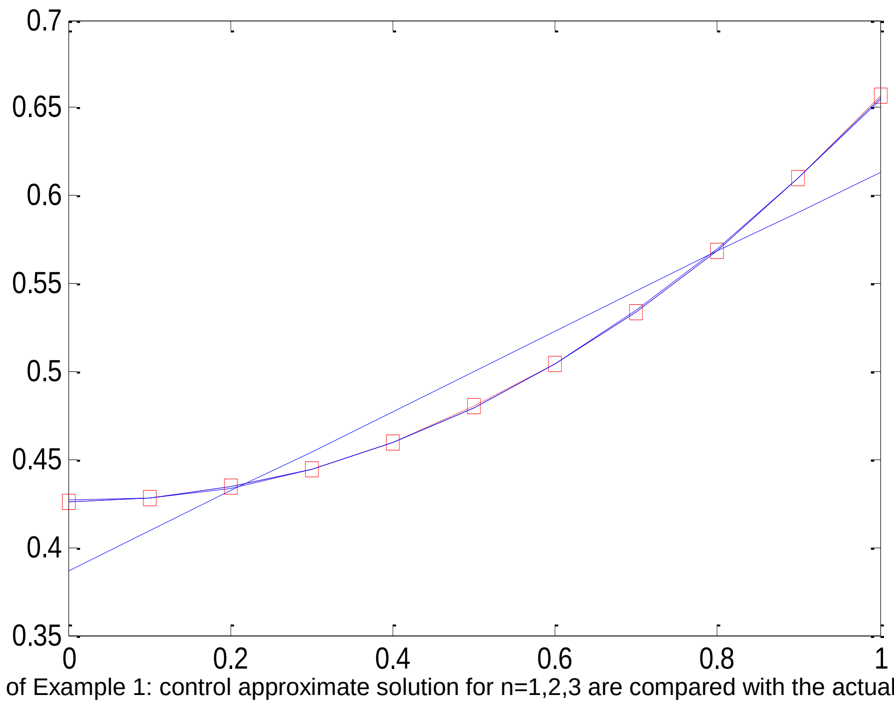

Figure 2. Solution of Example 1: control approximate solution for n=1,2,3 are compared with the actual analytical solution

Example 2

Consider the optimal control of a differential equation as below:

$$J = \frac{1}{2}\int_0^1 \left(U(\tau)^2 + X(\tau)^2\right) d\tau, \quad 0 \le \tau \le 1 \qquad (18)$$

The objective is to find the optimal state and optimal control based on minimizing the performance index. The state space in this problem is:

$$U(\tau) = \dot{X}(\tau) + X(\tau), \quad (19)$$

The boundary conditions can be considered as below:

$$X(0) = 1, \quad (20)$$

The exact solution for this problem can be achieved by Pontryagin's maximum principle as below:

$$X(\tau) = Ae^{\sqrt{2}t} + (1-A)e^{-\sqrt{2}t}, \quad (21)$$

$$U(\tau) = A(\sqrt{2}+1)e^{\sqrt{2}t} - (1-A)(\sqrt{2}-1)e^{-\sqrt{2}t}, \quad (22)$$

In this problem, *A* defines as below:

$$A = \frac{2\sqrt{2}-3}{2\sqrt{2}-(e^{\sqrt{2}})^2-3} \quad (23)$$

Table 2: Estimated and exact values of *PI* for Example 2.

Iteration	Proposed Method	Error	Mehne Method [16]	Error	Kafash Method [1]	Error
1	0.1943	1.16e^{-3}	0.2514	5.8e^{-2}	0.1943	1.3e^{-3}
2	0.19290	1.09 e^{-7}	0.1942	1.3e^{-3}	0.1929	2.2e^{-5}
3	0.1929	1.23 e^{-7}	0.1937	9.1e^{-4}	0.1929	4.7e^{-7}

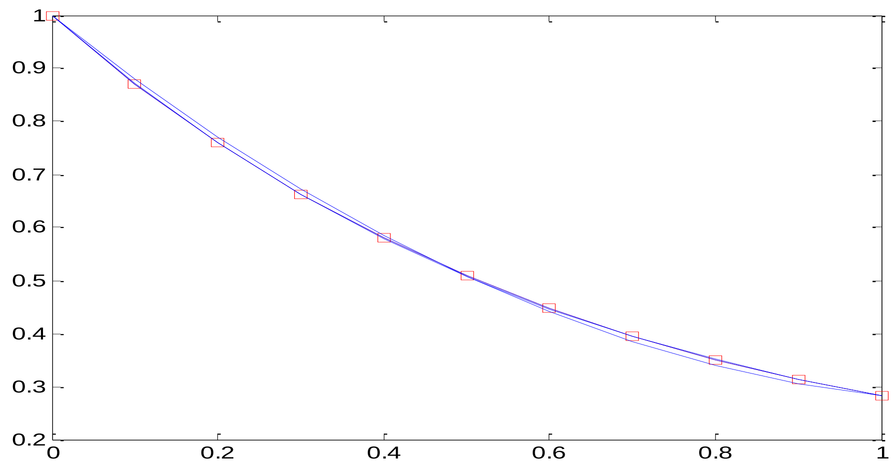

Figure 3. Solution of Example 2: state approximate solution for n=1,2,3 (blue lines) are compared with the actual analytical solution (red square)

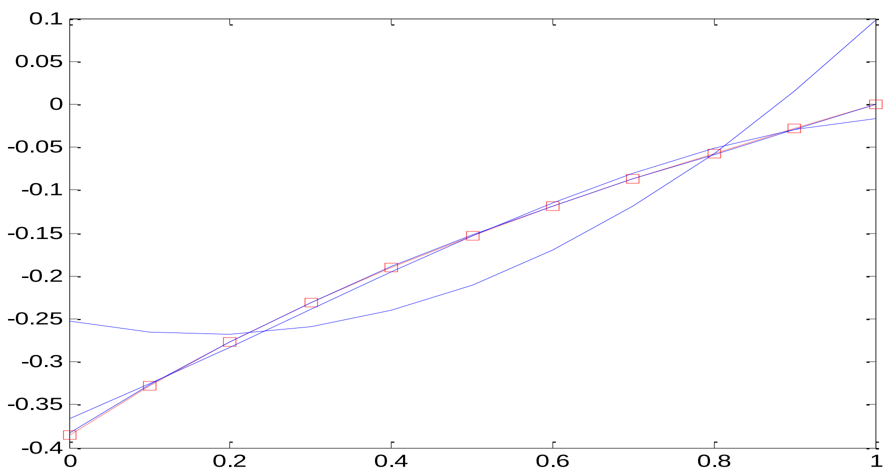

Figure 4. Solution of Example 2: control approximate solution for n=1,2,3 (blue lines) are compared with the actual analytical solution (red square)

CONCLUSION

A numerical method is presented for solving the optimal control problems. The solution is performed based on state parametrization. We used two numerical examples for evaluating the proposed technique accuracy and the results show a good performance for the presented approach. The considered method is based on combining the Chebyshev and wavelet polynomials for improving the polynomial accuracy. In simple terms, the proposed direct method in here has potential for achieving the continuous state and control variables as functions of time. The numerical value of the performance index is also achieved readily.

REFERENCES

Bertsekas, Dimitri P., et al. Dynamic programming and optimal control. Vol. 1. No. 2. Belmont, MA: Athena Scientific, 1995.

Chen, C. F., & Hsiao, C. H. (1975). AWalsh series direct method for solving variational problems. Journal of the Franklin Institute, 300(4), 265–280.

Cheng, B.,&Hsu, N. S. (1982). Analysis and parameter estimation of bilinear systems via block pulse functions. International Journal of Control, 36(1), 53–65.

Chou, J. H.,&Horng, I. R. (1985), Application of Chebyshev polynomials to the optimal control of time-varying linear systems, International Journal of Control, 41(1), 135–144.

Clement, Preston R. "Laguerre functions in signal analysis and parameter identification." Journal of the Franklin Institute 313.2 (1982): 85-95.

Heydari, M. H., et al. "Wavelets method for solving fractional optimal control problems." Applied Mathematics and Computation 286 (2016): 139-154.

M. Razzaghi, A. Arabshahi, “Optimal control of linear distributed parameter systems via polynomial” series, Inernat. J. Systems Sci. 20(1989) 1141-1148

Pantelev, A.B., Bortakovski, A.C. and Letova, T.A. (1996). *Some Issues and Examples in Optimal Control*, MAI Press, Moscow (in Russian).

Pontryagin, Lev Semenovich. Mathematical theory of optimal processes. CRC Press, 1987.

Razmjooy, Navid, and Mehdi Ramezani. "Analytical Solution for Optimal Control by the Second kind Chebyshev Polynomials Expansion." Iranian Journal of Science and Technology (Sciences) (2016).

Xu, Kuan. "The Chebyshev points of the first kind." Applied Numerical Mathematics 102 (2016): 17-30.